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COMPUTATIONAL HOMOGENIZATION OF ACOUSTIC PROBLEM IN PERFORATED PLATES

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Abstract: *We consider acoustic wave propagation described by Helmholtz equation and involving homogenized transmission conditions imposed along a thin perforated interface separating two halfspaces occupied by the acoustic medium. The homogenized transmission conditions are imposed on this perforated interface. The transmission conditions were obtained as the two-scale homogenization limit of the standard acoustic problem imposed in the layer perforated by a sieve-like obstacle with periodic structure. By using the sensitivity analysis we can solve the problem of an optimal design of the perforation to minimize the transmission loss in a domain embedding the interface. The perforated periodic structure is represented by a reference computational cell, whereby its geometry is controlled by the spline functions.*

Keywords: ***linear acoustics, homogenization, sensitivity analysis, transmission condition***

1. Introduction

Optimization of noise transmission in the acoustic fluid belongs to important merits of the acoustic engineering. Sieve-like structures are classical elements employed in noise-reducing devices. For example, in the exhaust silencers of the combustion engines the gas flows through ducts equipped with various sieves which in part may influence the transmission losses associated with acoustic waves propagating in the exhaust gas. In aerospace and automotive industry there are many applications related to acoustic waves and fluid flow where optimal design of the sieves (perforated slabs) is a challenging problem.

In the paper we deal with the acoustic transmission through a *perforated interface*, cf. Chen (1996); Bonnet-Bendhia and others (2005). The transmission conditions to be imposed on the interface plane were derived in Rohan and Lukeš (2010), using the asymptotic analysis. The limit model of an interface plane involves some homogenized impedance coefficients depending on the so-called microscopic problems; these are imposed in the *reference periodic cell* embedding an obstacle which represents the perforation. The two-scale modeling approach allows for an efficient treatment of complicated designs of perforations. The limit model was subjected to the sensitivity analysis in Rohan and Lukeš (2009). It resulted in the sensitivity formulas for the homogenized coefficients and we obtained the total variation of an objective function depending on the acoustic pressure w.r.t. the obstacle shape at the "microlevel".

An abstract optimization problem is formulated at three levels: at the "global" one the pressure field is controlled by an interface variable – the transversal acoustic momentum involved in the homogenized transmission condition; at the "homogenized interface" level, the interface variables are satisfy the non-local transmission conditions depending on the homogenized impedance parameters; finally, at the "microscopic level" these impedance (homogenized) parameters depend on solutions of auxiliary local problems featured by the shape of perforations.

2. Acoustic transmission through perforated interfaces

We consider the global problem of the wave propagation in a duct $\Omega \subset \mathbb{R}^3$ filled by the acoustic fluid. Ω is subdivided by perforated plane Γ_0 in two disjoint subdomains Ω^+ and Ω^-, so that $\Omega = \Omega^+ \cup$

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$\Omega^- \cup \Gamma_0$, see Figure 1 (obviously, much more general setting is possible). The acoustic pressure field p is discontinuous in general along Γ_0. In a case of no convection flow (the linear acoustics), the waves propagating in Ω are described by the following equations where κ is the wave number (i.e. frequency $\omega = \kappa c$)

$$
\begin{aligned}
\nabla^2 p + \kappa^2 p = 0 \quad & \text{in } \Omega^+ \cup \Omega^- , \\
\text{transmission conditions} \quad \mathcal{G}(\kappa, [p]_-^+, [\partial p/\partial n]_-^+) = 0 \quad & \text{on } \Gamma_0 , \\
r \mathrm{i} \kappa p + \frac{\partial p}{\partial n} = s 2 \mathrm{i} \kappa \bar{p} \quad & \text{on } \partial\Omega ,
\end{aligned}
\tag{1}
$$

where s, r and $\bar{p}$ are given data, $[\cdot]_-^+$ is the jump across Γ_0. $\frac{\partial p}{\partial n} = \boldsymbol{n} \cdot \nabla p$ is the normal derivative on Γ_0. The homogenized transmission conditions $\mathcal{G} = 0$ developed in Rohan and Lukeš (2010) introduce two internal variables on Γ_0: the "in-layer" acoustic potential p^0 and the "trans-layer" acoustic velocity g^0, which is coupled with the "off-layer" fields through: $\partial p/\partial n^\pm = \pm \mathrm{i}\kappa g^0$, so that $[\partial p/\partial n]_-^+ = 0$. Boundary $\partial\Omega = \Gamma_w \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ of the duct is split into walls and the input/output parts; by the constants r, s in $(1)_3$ different conditions on $\partial\Omega$ are respected: $r = s = 0$ on the duct walls Γ_w, whereas $r = s = 1$ on Γ_{in} and $r = 1$, $s = 0$ on Γ_{out}.

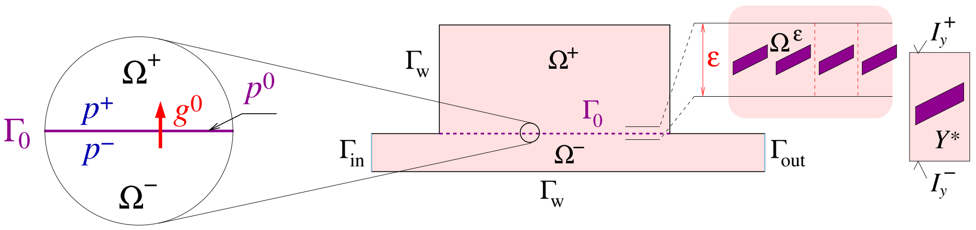


Fig. 1: Left: illustration of the transmission coupling – the acoustic pressure jump is proportional to the transverse acoustic velocity g^0. Center: the domain and boundary decomposition of the global acoustic problem considered. Right: perforated interface and the representative periodic cell $Y = Y^* \cup \overline{S}$.

3. Acoustic problem with homogenized sieve

We now formulate the *state problem* describing acoustic waves in open bounded domain Ω with immersed homogenized sieve represented by non-local transmission conditions. We need the following notation:

$$
a_\Omega(p, q) = \int_\Omega \nabla p \cdot \nabla q , \quad (p, q)_\Omega = \int_\Omega pq , \quad \langle p, q \rangle_{\Gamma_0} = \int_{\Gamma_0} pq .
$$

The problem is defined at two levels:

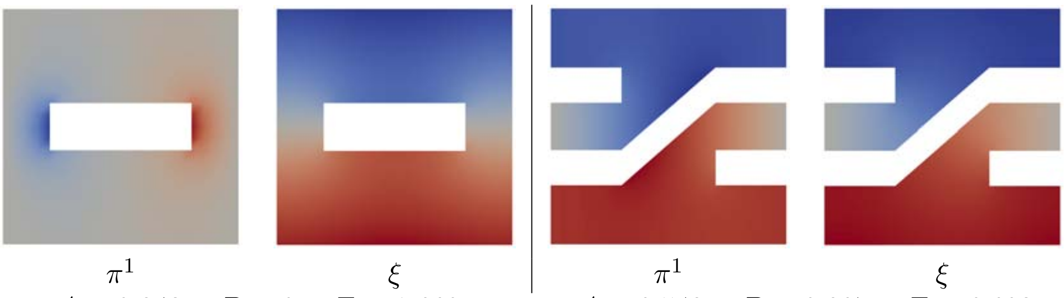
At the *global level* the interface conditions involve 3 geometrical parameters A, B, F which characterize the design of the sieve perforation; we define (summation $\alpha, \beta = 1, 2$)

$$
\mathcal{A}(p, q) = \int_{\Gamma_0} A_{\alpha\beta} \partial_\beta p \partial_\alpha q , \quad \mathcal{B}(g, q) = \int_{\Gamma_0} B_\alpha g \partial_\alpha q , \quad \mathcal{F}(g, h) = \int_{\Gamma_0} F g h . \tag{2}
$$

The *global* problem is to find $(p.p^0, g^0) \in H^1(\Omega \setminus \Gamma_0) \times H^1(\Gamma_0) \times L^2(\Gamma_0)$ such that

$$
\begin{aligned}
a_\Omega(p, q) - \kappa^2 (p, q)_\Omega + \mathrm{i}\kappa \langle p, q \rangle_{\Gamma_{\text{in-out}}} - \mathrm{i}\kappa \langle g^0, [q]_-^+ \rangle_{\Gamma_0} &= 2\mathrm{i}\kappa \langle \bar{p}, q \rangle_{\Gamma_{\text{in}}} \\
\mathcal{A}(p^0, \phi) - \kappa^2 \varsigma^* \langle p^0, \phi \rangle_{\Gamma_0} + \mathrm{i}\kappa \mathcal{B}(g^0, \phi) &= 0 , \\
-\mathrm{i}\kappa \mathcal{B}(\psi, p^0) - \kappa^2 \mathcal{F}(g^0, \psi) + \mathrm{i}\kappa \frac{1}{\varepsilon^0} \langle [p]_-^+, \psi \rangle_{\Gamma_0} &= 0 ,
\end{aligned}
\tag{3}
$$

for all $(q, \phi, \psi) \in H^1(\Omega \setminus \Gamma_0) \times H^1(\Gamma_0) \times L^2(\Gamma_0)$, where ε_0 is the real thickness of the layer.



π^1 $\quad$ ξ $\quad$ π^1 $\quad$ ξ

$A = 0.843, \quad B = 0, \quad F = 1.660$ $\qquad$ $A = 0.542, \quad B = 0.897, \quad F = 6.603$

Tab. 1: Correctors π, ξ and homogenized coefficients for two different geometrical structures.

At the *local* level the geometrical parameters A, B, F are determined upon solving "microscopic problems". The perforation design is characterized by computational cell $Y = \Xi \times]-1/2, +1/2[$ with $\Xi =]0, b_1[\times]0, b_2[$, where the fluid occupies domain Y^* and $S = Y \setminus \overline{Y^*}$ represents a rigid obstacle, see Figure 1. Further $I_y^{\pm} = \Xi \pm (0, 0, 1)$ are the "lower" and "upper" faces of Y. In (3), $\varsigma^* = |Y^*|/|\Xi|$ is the porosity. Below the space $H^1_{\#}(Y^*)$ contains all Ξ-periodic functions in the Sobolev space $H^1(Y^*)$. The local problems read: find $\pi^\beta, \xi \in H^1_{\#}(Y^*)$ such that

$$
\begin{aligned}
\left(\nabla_y \pi^\beta, \nabla_y \psi\right)_{Y^*} &= -\int_{Y^*} \partial^y_\beta \psi\,, \quad \beta = 1, 2\,, \\
(\nabla_y \xi, \nabla_y \psi)_{Y^*} &= -\left(\int_{I_y^+} \psi - \int_{I_y^-} \psi\right)\,,
\end{aligned}
\tag{4}
$$

for all $\psi \in H^1_{\#}(Y^*)$, where $\nabla_y = (\partial/\partial y_\beta)$ and $(,)_{Y^*}$ is the inner product in $L^2(Y^*)$. Using the local responses, the geometrical parameters can now be computed, see (2) and Figure 1:

$$
\begin{aligned}
A_{\alpha\beta} &= \frac{1}{|\Xi|}\left(\nabla_y(\pi^\beta + y_\beta), \nabla_y(\pi^\alpha + y_\alpha)\right)_{Y^*}\,, \\
B_\alpha &= \fint_{Y^*} \partial^y_\alpha \xi = \fint_{I_y^+} \pi^\alpha - \fint_{I_y^-} \pi^\alpha\,, \qquad F = -\fint_{I_y^+} \xi - \fint_{I_y^-} \xi\,,
\end{aligned}
\tag{5}
$$

where $\fint = |\Xi|^{-1} \int$. Note $F > 0$ and A is positive definite.

Table 1 illustrates how the homogenized coefficients and corrector functions depend on the geometrical arrangement of the reference cell Y.

4. Optimal design problem

One of the most frequently used criteria of optimality in acoustics is related to transmission loss (TL) evaluated using two pressures $p^{in} = p$ on Γ_{in}, $p^{out} = p$ on Γ_{out}, where p satisfies the *state problem* (3). In our numerical tests we observed some remarkable sensitivity of TL on the perforation design, see Rohan and Lukeš (2010). In the further sections we will employ the following objective function:

$$
\Phi_{\mathrm{TL}}(p) = \hat{\Phi}(p^{in}, p^{out}) = 20\,log\left(\frac{|p^{in}|}{|p^{out}|}\right) - \widetilde{TL}. \tag{6}
$$

Let the perforation design be controlled by design variables $\mathbf{d}$ which describe the shape of obstacle S and, thereby, the shape of domain Y^*, so that $\mathbf{d}$ influences the homogenized coefficients A, B, F involved in (3). Let us recall that these coefficients are integrals of functions π^β, ξ which are solutions of the microscopic problems (4) posed in Y^*. At the global level, $\mathbf{d}$ influences the overall acoustic fields (p, p^0, g^0).

We can now define the *optimal perforation design problem*:

$$\min_{\mathbf{d}\in D_{adm}} \Phi(p,p^0,g^0) \quad \text{subject to:} \quad (p,p^0,g^0) \text{ solves (3), where } A,B,F \text{ are given by (4),(5),} \tag{7}$$

where D_{adm} is the set of admissible designs, constraining shape regularity of ∂S and typically some other features, like the size of the obstacle (thickness), or porosity of the interface.

To solve (7) using gradient-based methods, the sensitivity of Φ w.r.t. the design $\mathbf{d} = (d_i)$ must be supplied at any iteration (Φ can be substituted by $\Phi_{\rm TL}$ or $-\Phi_0$, for instance). For this, any component d_i is associated with the *design velocity* field $\vec{\mathcal{V}}^i$ which can be constructed e.g. by solving an auxiliary elasticity problem in domain Y^* or it results from derivative of the spline-based parametrization of the reference cell mesh, see Fig. 2. The shape sensitivities $\delta A_{\alpha\beta}(\vec{\mathcal{V}}^i), \delta B_\beta(\vec{\mathcal{V}}^i), \delta F(\vec{\mathcal{V}}^i)$ and $\delta\zeta^*(\vec{\mathcal{V}}^i)$ of coefficients $A_{\alpha\beta}, B_\beta, F$ and ζ^* can be obtained, as described in Rohan and Lukeš (2009), using the general approach based on the material derivative.

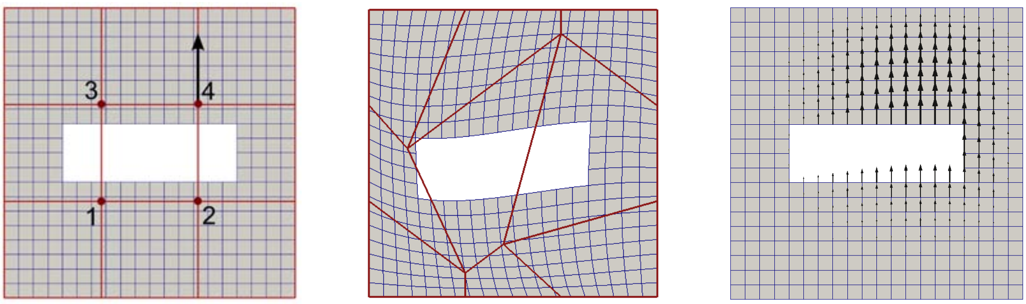
The total design sensitivity $\delta\Phi(p,p^0,g^0;\vec{\mathcal{V}}^i) = \frac{\partial}{\partial d_i}\Phi$ is obtained by formula

$$\begin{aligned}\delta\Phi(p,p^0,g^0;\vec{\mathcal{V}}^i) = 2\Re\Big\{ &\int_{\Gamma_0} \delta A_{\alpha\beta}(\vec{\mathcal{V}}^i)\partial_\beta p^0 \partial\tilde{p}^0 - \kappa^2 \int_{\Gamma_0} \delta F(\vec{\mathcal{V}}^i) g^0 \tilde{g}^0 \\ &-\kappa^2 \delta\zeta^*(\vec{\mathcal{V}}^i) \int_{\Gamma_0} p^0 \tilde{p}^0 + \mathrm{i}\kappa \int_{\Gamma_0} \delta B_\alpha(\vec{\mathcal{V}}^i)\left(\partial_\alpha \tilde{p}^0\, g^0 - \partial_\alpha p^0\, \tilde{g}^0\right)\Big\}\ ,\end{aligned} \tag{8}$$

where $(\tilde{p},\tilde{p}^0,\tilde{g}^0) \in H^1(\Omega\setminus\Gamma_0)\times H^1(\Gamma_0)\times L^2(\Gamma_0)$ is the *adjoint state*, cf. Feijóo and others (2004), satisfying the *adjoint equation*, see Rohan and Lukeš (2009) for details,

$$\begin{aligned} &a_\Omega(\tilde{p},\,q) - \kappa^2(\tilde{p},\,q)_\Omega + \mathrm{i}\kappa\langle\tilde{p},\,q\rangle_{\Gamma_{\rm in-out}} - \mathrm{i}\kappa\left\langle\psi,\,[\tilde{p}]^+_-\right\rangle_{\Gamma_0} + \mathrm{i}\kappa\frac{1}{\varepsilon^0}\left\langle[q]^+_-,\,\tilde{g}^0\right\rangle_{\Gamma_0} \\ &+\mathcal{A}(\tilde{p}^0,\phi) - \kappa^2\varsigma^*\left\langle\tilde{p}^0,\,\phi\right\rangle_{\Gamma_0} + \mathrm{i}\kappa\mathcal{B}(\psi,\tilde{p}) - \mathrm{i}\kappa\mathcal{B}(\tilde{g}^0,\phi) - \kappa^2\mathcal{F}(\tilde{g}^0,\psi) \\ &= -\frac{1}{2}\left(\partial_{\Re(p,p^0,g^0)}\Phi(p,p^0,g^0;q,\phi,\psi) - \mathrm{i}\partial_{\Im(p,p^0,g^0)}\Phi(p,p^0,g^0;q,\phi,\psi)\right)\ ,\end{aligned} \tag{9}$$

for all $(q,\phi,\psi) \in H^1(\Omega\setminus\Gamma_0)\times H^1(\Gamma_0)\times L^2(\Gamma_0)$, where $\Re$ and $\Im$ is the real and the imaginary part, respectively.



(A) (B) (C)

Fig. 2: (A): spline-based parametrization of the reference cell mesh, initial state; (B): FE mesh modified by moving position of the inner control points; (C): design velocity field $\vec{\mathcal{V}}$ associated with the shape perturbation – y-shift of control point 4.

4.1. Numerical example

We consider a 2D problem of acoustic waves in a waveguide equipped with a perforated plate, see Fig. 3, designed by repeating a reference cell which geometry is controlled by the spline functions. In the optimal design problem we allow four inner control points to move, see Fig. 2, so we have eight optimization parameters (two coordinates for each control point). The objective function to be optimized

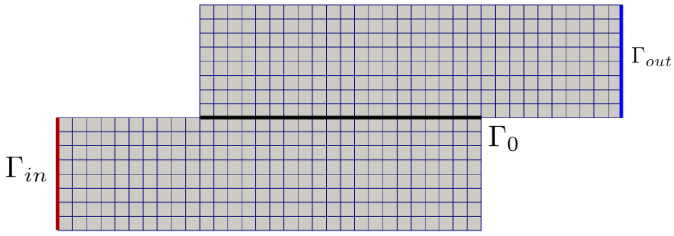
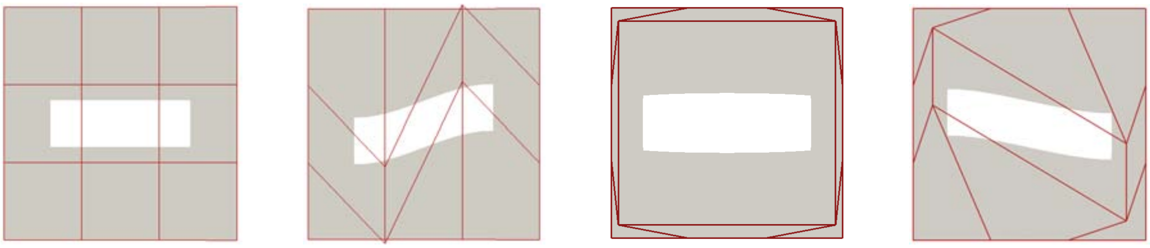


Fig. 3: Acoustic waveguide equipped with a perforated plate Γ_0.

is considered in such form to find a shape of the obstacle for which the transmission loss $TL(p)$ is close to a required value $\widetilde{TL}$. The results (local minima) were obtained by the SQP algorithm with box constraints which secure the "mesh deformation" during the design iterations.

We started the optimization with two different initial states given by the parametrization vectors $[0, 0, 0, 0, 0, 0, 0, 0]$ (for rectangular shape), $[0, 0, 0, 0, -0.3, 0.3, -0.3, 0.3]$ (for distorted shape), see Fig. 4, (A) and (B). In both cases, the optimization process resulted in the shape parametrized by the vector $[-0.304, 0.304, -0.304, 0.304, -0.276, -0.276, 0.276, 0.276]$, the final shape is depicted in Fig. 4(C). The box constraints were chosen as $\langle -0.35, 0.35\rangle$ for all eight parameters to secure the "safe mesh deformation". Figure 4(D) shows the shape ($[-0.25, 0.25, -0.25, 0.25, 0.25, -0.25, 0.25, -0.25]$) of the reference cell for which the transmission loss $\widetilde{TL}$ was computed. The fact, that the optimization finished in state Fig. 4(C) and not in Fig. 4(D), can be explained by the existence of multiple local minima of the used objective function.



(A) (B) (C) (D)

Fig. 4: (A) and (B): two different initial states used in optimization, parametrization: $[0, 0, 0, 0, 0, 0, 0, 0]$ *(left),* $[0, 0, 0, 0, -0.3, 0.3, -0.3, 0.3]$ *(middle); (C): final shape of the obstacle after optimization, param.:* $[-0.304, 0.304, -0.304, 0.304, -0.276, -0.276, 0.276, 0.276]$ *(D): the shape with param.* $[-0.25, 0.25, -0.25, 0.25, 0.25, -0.25, 0.25, -0.25]$ *for which* $\widetilde{TL}$ *was computed.*

5. Conclusion

The "multi-scale" homogenization approach is employed for an efficient treatment of the optimal perforation design. We use the spline parametrization to control the shape of the solid obstacle forming the perforation. The model and its sensitivity discussed in this paper are implemented in our in-house developed finite element based code *SfePy* (Cimrman and others (2012)). The numerical example demonstrate the ability of the optimization method to find an appropriate shape of the solid obstacle for a given transmission loss value.

Acknowledgments

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