

NUMERICAL SIMULATION OF TRANSITIONAL FLOWS WITH LAMINAR KINETIC ENERGY

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Abstract: *The article deals with the numerical solution of transitional flows. The single point $k-k_L-\omega$ model of Walters and Cokljat (2008) based on the use of a laminar kinetic energy transport equation is considered. The model doesn't require to evaluate integral boundary layer parameters (e.g. boundary layer thickness) and is therefore suitable for implementation into codes working with general unstructured meshes. The performance of the model has been tested for the case of flows over a flat plate with zero and non-zero pressure gradients. The results obtained with our implementation of the model are compared to the experimental data of ERCOFTAC.*

Keywords: *Turbulence, transition, Navier–Stokes equations*

1. Introduction

The laminar-turbulent transition plays very important role in many flows of engineering interest. It has big impact on the heat transfer and losses. Unfortunately most of the state-of-art turbulence models (e.g. Menters SST $k - \omega$ model, EARSM model of Hellsten) completely fail with the prediction of transition. However there are some attempts to modify basic models (e.g. low Reynolds model of Wilcox (1998), TSL model of Zheng et al. (1998)) with promising results, the experience shows that this approach is not capable of reliably capturing all factors that affect transition, see Menter et al. (2006).

The algebraic models based on empirical correlations (see e.g, Straka and Příhoda (2010)) offers simple approach with sufficient accuracy. On the other hand the implementation into a general unstructured code is quite difficult due to necessity of some non-local information (momentum boundary layer thickness, intermittency at wall, etc.). Therefore the applicability of these models is more-less limited to research/academic codes using structured meshes.

This article deals with the RANS-based transitional model developed by Walters and Cokljat (2008). The three-equation model is based on the low Reynolds $k - \omega$ model with an equation for the so called laminar kinetic energy k_L expressing the energy of stream-wise fluctuations in pre-transitional region. The main advantage of the model is its local formulation, it means that it can be easily implemented into unstructured solvers. Moreover it can (at least in principle) handle flows in complex geometry. Unfortunately the description of the model in Walters and Cokljat (2008) contains some errors (probably typos) which lead to strong underestimation of the friction in turbulent region. The aim of the article is to show the correct version of the model and to test the performance of the model for simple flows over flat plate.

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2. Mathematical model

2.1. Navier–Stokes equations

The viscous compressible flows is described by the set of Favre–averaged Navier–Stokes equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0, \quad (1)$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} + \frac{\partial p}{\partial x_i} = \frac{\partial(t_{ij} + \tau_{ij})}{\partial x_j}, \quad (2)$$

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial[(\rho E + p)u_j]}{\partial x_j} = \frac{\partial}{\partial x_j} \left[u_i(t_{ij} + \tau_{ij}) + \left(\frac{\mu}{Pr} + \rho \alpha_\theta \right) \frac{\partial h}{\partial x_j} \right], \quad (3)$$

where ρ is the density, u_i are the components of the velocity vector, p is the static pressure, E is the specific total energy, $h = E + p - u_i u_i / 2$ is the specific enthalpy, t_{ij} is the mean viscous stress tensor, $\tau_{ij} = -\overline{\rho u'_i u'_j}$ is the Reynolds stress tensor, μ is the viscosity, Pr is the Prandtl number, and α_θ is the turbulent thermal diffusivity.

We assume perfect gas (the air) with $p = (\kappa - 1)(\rho E - \rho u_i u_i / 2)$ where $\kappa = 1.4$ is the constant specific heat ratio. The flow is Newtonian with constant viscosity μ , hence $t_{ij} = 2\mu(S_{ij} - \frac{1}{3}S_{ll}\delta_{ij})$ where $S_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i) / 2$.

2.2. Turbulence model

The turbulence model is based on the Boussinesq hypothesis

$$\tau_{ij} = 2\rho\nu_T(S_{ij} - \frac{1}{3}S_{ll}\delta_{ij}) - \frac{1}{3}\rho k\delta_{ij}, \quad (4)$$

where ν_T is the turbulent kinematic viscosity and k is the turbulent kinetic energy.

We assume a three equation model of Walters and Cokljat (2008) with the transport equations for the turbulent kinetic energy k_T , the laminar kinetic energy k_L , and the turbulent frequency ω . The equations are

$$\frac{D(\rho k_T)}{Dt} = \rho(P_{k_T} + R_{BP} + R_{NAT} - \omega k_T - D_T) + \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\rho \alpha_T}{\sigma_k} \right) \frac{\partial k_T}{\partial x_j} \right], \quad (5)$$

$$\frac{D(\rho k_L)}{Dt} = \rho(P_{k_L} - R_{BP} - R_{NAT} - D_L) + \frac{\partial}{\partial x_j} \left[\mu \frac{\partial k_L}{\partial x_j} \right], \quad (6)$$

$$\begin{aligned} \frac{D(\rho \omega)}{Dt} = & \rho \left[C_{\omega 1} \frac{\omega}{k_T} P_{k_T} + \left(\frac{C_{\omega R}}{f_W} - 1 \right) \frac{\omega}{k_T} (R_{BP} + R_{NAT}) - C_{\omega 2} \omega^2 \right. \\ & \left. + C_{\omega 3} f_\omega \alpha_T f_W^2 \frac{\sqrt{k_T}}{d^3} \right] + \frac{\partial}{\partial x_j} \left[\left(\mu + \frac{\rho \alpha_T}{\sigma_\omega} \right) \frac{\partial \omega}{\partial x_j} \right]. \end{aligned} \quad (7)$$

The various terms in the equations represents production, destruction, transport, and diffusion. However the structure of the model is more-less clear, there is a confusion in the definition of individual terms in the literature. The original Walters and Leylek's model (see Walters and Leylek (2004)) uses $k_T - k_L - \epsilon$ formulation. The model was later re-formulated using $k_T - k_L - \omega$ (see Walters and Leylek (2005) or Holloway et al. (2004)) and the current version was published in Walters and Cokljat (2008). Unfortunately it seems that the last article contains some errors. Therefore we will write here all individual terms and we will comment the differences of our version with respect to the original article.

The production of turbulent and laminar kinetic energy is

$$P_{k_T} = \nu_{T,s} S^2, \quad (8)$$

$$P_{k_L} = \nu_{T,l} S^2, \quad (9)$$

where $S = \sqrt{2S_{ij}S_{ij}}$. The “small-scale” eddy viscosity is defined as

$$\nu_{T,s} = f_\nu f_{INT} C_\mu \sqrt{k_{T,s}} \lambda_{eff}, \quad (10)$$

where $k_{T,s}$ is the effective small-scale turbulence

$$k_{T,s} = f_{SS} f_W k_T. \quad (11)$$

The wall-limited turbulence length scale λ_{eff} and damping function f_W is

$$\lambda_{eff} = \min(C_\lambda d, \lambda_T), \quad (12)$$

$$\lambda_T = \sqrt{k_T \omega}, \quad (13)$$

$$f_W = \left(\frac{\lambda_{eff}}{\lambda_T} \right)^{\frac{2}{3}}, \quad (14)$$

here d is the wall distance. Note that the article Walters and Cokljat (2008) does not include the exponent $2/3$ in the definition of f_W . The original model Walters and Leylek (2004) as well as the Walters and Leylek (2005) do include the exponent. The origins of the $2/3$ exponent come from the $k - \epsilon$ formulation where the turbulent integral length scale is

$$\lambda_T = \frac{k_T^{\frac{3}{2}}}{\epsilon}, \quad (15)$$

therefore f_W actually limits the length scale to λ_{eff} .

The following terms are according to Walters and Cokljat (2008)

$$f_\nu = 1 - \exp\left(-\frac{\sqrt{Re_T}}{A_\nu}\right), \quad (16)$$

$$Re_T = \frac{f_W^2 k_T}{\nu \omega}, \quad (17)$$

$$f_{SS} = \exp\left[-\left(\frac{C_{ss} \nu \Omega}{k_T}\right)^2\right], \quad (18)$$

$$C_\mu = \frac{1}{A_0 + A_S \left(\frac{S}{\omega}\right)}. \quad (19)$$

The intermittency factor f_{INT} is

$$f_{INT} = \min\left(\frac{k_T}{C_{INT}(k_T + k_L)}, 1\right). \quad (20)$$

Note that the factor f_{INT} is defined with k_L in nominator in Walters and Cokljat (2008), but the article Walters and Leylek (2005) gives correct form with k_T .

The production of laminar kinetic energy k_L is assumed to be given by large-scale near wall turbulence

$$k_{T,l} = k_T - k_{T,s}. \quad (21)$$

The production term is then given by the equation (9) where

$$\nu_{T,l} = \min\left\{f_{\tau,l} C_{l1} \frac{\Omega \lambda_{eff}^2}{\nu} \sqrt{k_{T,l} \lambda_{eff}} + \beta_{TS} C_{l2} Re_\Omega d^2 \Omega, \frac{k_L + k_{T,l}}{2S}\right\}. \quad (22)$$

Here

$$Re_\Omega = \frac{d^2 \Omega}{\nu}, \quad (23)$$

$$\beta_{TS} = 1 - \exp\left(-\frac{\max(Re_\Omega - C_{TS,crit}, 0)^2}{A_{TS}}\right), \quad (24)$$

$$f_{\tau,l} = 1 - \exp\left(-C_{\tau,l} \frac{k_{tau,l}}{\lambda_{eff}^2 \Omega^2}\right). \quad (25)$$

The dissipation ϵ_{Tot} is divided to an isotropic ($k_T\omega$) and anisotropic ($D_{T/L}$) part (similarly as in the low Reynolds Launder and Sharma $k - \epsilon$ model) with

$$D_T = \nu \frac{\partial \sqrt{k_T}}{\partial x_i} \frac{\partial \sqrt{k_T}}{\partial x_i}, \quad (26)$$

$$D_L = \nu \frac{\partial \sqrt{k_L}}{\partial x_i} \frac{\partial \sqrt{k_L}}{\partial x_i}. \quad (27)$$

However the the balance between the dissipation ϵ_{Tot} and diffusion $\frac{\partial}{\partial y} \left(\nu \frac{\partial k_{T/L}}{\partial y} \right)$ in the laminar sublayer suggests the same formula multiplied by 2 (see e.g. Launder-Sharma $k - \epsilon$ model or the older versions of $k - k_L - \epsilon$ and $k - k_l - \omega$) model, the above mentioned form was proposed in the new model and used in our calculations.

The turbulent diffusivity α_T is

$$\alpha_T = f_\nu C_{\mu, std} \sqrt{k_{T,s}} \lambda_{eff}, \quad (28)$$

and the damping function f_ω is

$$f_\omega = 1 - \exp \left[-0.41 \left(\frac{\lambda_{eff}}{\lambda_T} \right)^4 \right]. \quad (29)$$

The remaining terms R_{BP} and R_{NAT} express the laminar-turbulent transition in terms of the energy transfer from k_L to k_T . They are of the form

$$R_{BP} = C_R \beta_{BP} k_L \omega / f_W, \quad (30)$$

$$R_{NAT} = C_{R, NAT} \beta_{NAT} k_L \Omega. \quad (31)$$

The bypass transition is driven by the β_{BP} function

$$\beta_{BP} = 1 - \exp \left(-\frac{\phi_{BP}}{A_{BP}} \right), \quad (32)$$

$$\phi_{BP} = \max \left(\frac{k_T}{\nu \Omega} - C_{BP, crit}, 0 \right), \quad (33)$$

and the natural transition by the β_{NAT} function

$$\beta_{NAT} = -\exp \left(-\frac{\phi_{NAT}}{A_{NAT}} \right), \quad (34)$$

$$\phi_{NAT} = \max \left(Re_\Omega - \frac{C_{NAT, crit}}{f_{NAT, crit}}, 0 \right), \quad (35)$$

$$f_{NAT, crit} = 1 - \exp \left(-C_{NC} \frac{\sqrt{k_L} d}{\nu} \right). \quad (36)$$

The turbulent kinematic viscosity used in the momentum equations is then

$$\nu_T = \nu_{T,s} + \nu_{T,l}. \quad (37)$$

The turbulent thermal diffusivity α_θ is then

$$\alpha_\theta = f_W \frac{k_T}{k_T + k_L} \frac{\nu_{T,s}}{Pr} + (1 - f_W) C_{\alpha, \theta} \sqrt{k_T} \lambda_{eff}. \quad (38)$$

The coefficient $C_{\omega 2} = 0.92$ is constant in the original article. Nevertheless the correct form is

$$C_{\omega 2} = 0.92 f_W^2. \quad (39)$$

case	U [m s ⁻¹]	k_T [m ² s ⁻²]	ω [s ⁻¹]	Tu [%]	$\mu_{T,std}/\mu$ [-]
T3A	5.4	0.047 63	23.8	3.30	12
T3B	9.2	1.128 27	56.8	9.43	120
T3A-	19.8	0.048 57	23.8	0.91	12
T3C2	5.5	0.055 58	35.0	3.50	10

Tab. 1: Inlet conditions for flat plate calculations at $x = -0.05$ m, here $\mu_{T,std} := C_{\mu,std}\rho k_T/\omega$.

The other constants are

$$\begin{array}{lll}
A_0 = 4.04 & C_{INT} = 0.75 & C_{\omega 1} = 0.44 \\
A_S = 2.12 & C_{TS,crit} = 1000 & C_{\omega 3} = 0.3 \\
A_\nu = 6.75 & C_{R,NAT} = 0.02 & C_{\omega R} = 1.5 \\
A_{BP} = 0.6 & C_{l1} = 3.4 \cdot 10^{-6} & C_\lambda = 2.495 \\
A_{NAT} = 200 & C_{l2} = 10^{-10} & C_{\mu,std} = 0.09 \\
A_{TS} = 200 & C_R = 0.12 & Pr = 0.85 \\
C_{BP,crit} = 1.2 & C_{\alpha,\theta} = 0.035 & \sigma_k = 1 \\
C_{NC} = 0.1 & C_{SS} = 1.5 & \sigma_\omega = 1.17 \\
C_{NAT,crit} = 1250 & C_{tau,l} = 4360 &
\end{array}$$

3. Simulation of flows over a flat plate

The model has been validated using T3 series of experimental flat plate test cases of ERCOFTAC. The T3A, T3B, and T3A- test cases had zero stream-wise pressure gradients with free-stream turbulence of 3%, 6%, and 1% respectively Coupland (1990a). The T3C2 has favorable pressure gradient in the first part of the plate followed by the adverse pressure gradient in the second part, see Coupland (1990b).

The calculation was carried out with OpenFOAM package with our implementation of the $k - k_L - \omega$ model. The numerical solution was obtained with finite volume method using SIMPLEC scheme for compressible flows (see eg. Ferziger and Perić (1999)).

The zero-pressure gradient cases (i.e. T3A, T3A-, and T3B) were calculated using a rectangular domain $\Omega = [-0.05, 2.9] \times [0, 0.175]$ m where the flat plate starts at $x = 0$ m. The mesh consists of 635×105 cells where 600 cells were at the plate and 35 cells in the inlet region. The mesh was refined in the vicinity of the inlet edge (see fig. 1) and in the wall normal direction with $y_1 \approx 10^{-5}$ m i.e. $y^+ \leq 1$.

The following boundary conditions were prescribed:

inlet: at the inlet plane ($x = -0.05$ m) we prescribe the velocity vector u_i , the temperature ($T = 293.15$ K), the turbulent kinetic energy k_T , the laminar kinetic energy $k_L = 0$ m² s⁻², and the specific dissipation rate ω . The pressure is calculated with the homogeneous Neumann condition $\partial p/\partial n = 0$.

wall: at the wall ($x = 0$ m to 2.9 m and $y = 0$ m) we prescribe non-slip condition for velocity ($u_i = 0$ m/s), the homogeneous Neumann condition for pressure $\partial p/\partial n = 0$, zero turbulent and laminar kinetic energy $k_T = k_L = 0$ m² s⁻², and the homogeneous Neumann condition for the specific dissipation rate $\partial \omega/\partial n = 0$.

outlet: at the outlet ($x = 2.9$ m) we prescribe the static pressure $p = 101$ kPa and we use homogeneous Neumann conditions for all remaining quantities.

symmetry: at the rest of the boundary (the upper boundary at $y = 0.175$ m and the lower boundary in front of the plate) we assume symmetry condition for all variables (i.e. the slip condition).

We use constant dynamic viscosity $\mu = 1.8 \times 10^{-5}$ Pa s and the ideal gas constant $R = 287$ J kg⁻¹ K⁻¹. The parameters of inlet flows are given in the table 1.

The figure 1 shows comparison of computed skin friction for T3A and T3B case with the experimental data of ERCOFTAC. One can see that the ready-made implementation of the model (labeled by OF

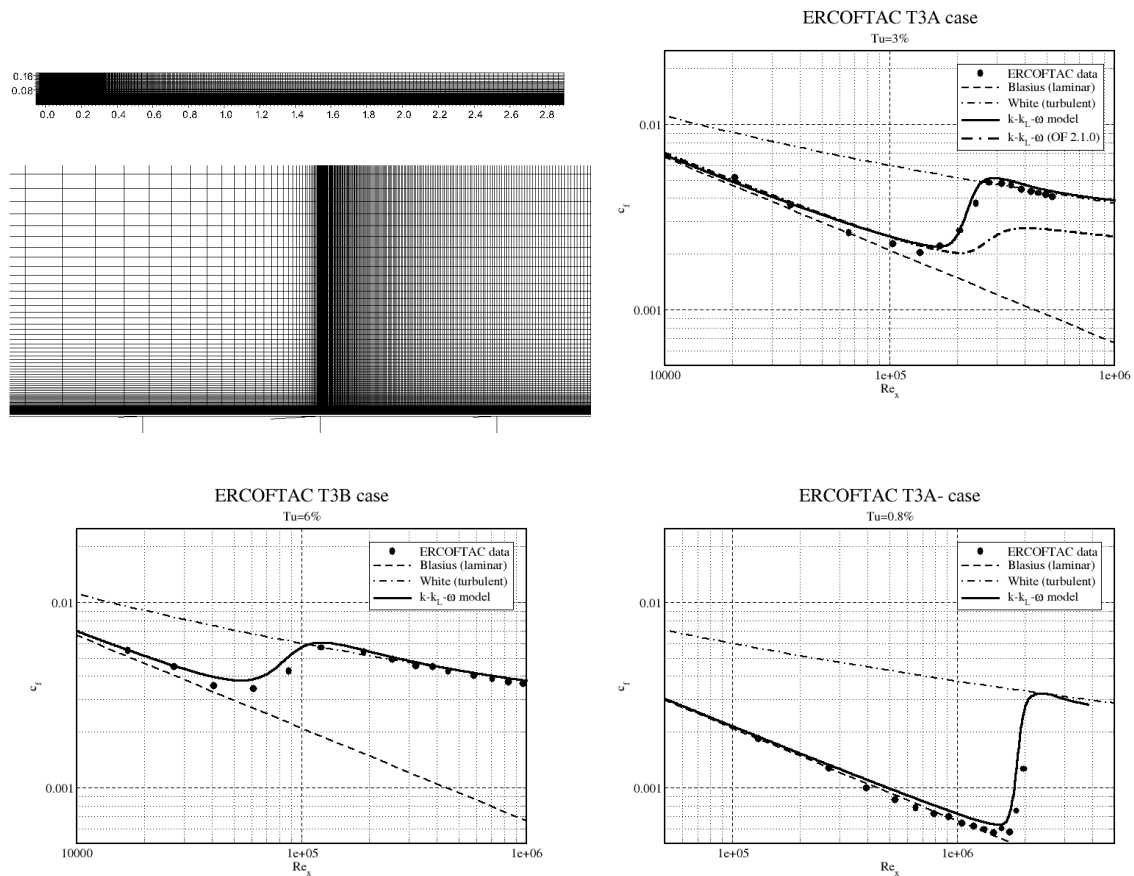


Fig. 1: Friction coefficient for zero-pressure gradient flows over a flat plate (ERCOFTAC T3A, T3B, T3A-cases).

2.1.0 at the figure) fails even with T3A case. On the other hand our implementation of corrected model gives quite good agreement with experimental data form ERCOFTAC database. The results show that the transition onset is very well captured in T3A and T3A- cases. In the T3B case with high turbulence intensity the transition seems to be shifted little bit upstream.

For the non-zero pressure gradient case (T3C2) we use

For the flow with pressure gradient (T3C2 case) we assume a domain with shaped upper boundary (see fig. 2). The shape was constructed in order to match the velocity distribution in the ERCOFTAC experiment. Fig. 2 shows the comparison of experimental data with the calculated velocity distribution at $y = 0.05$ m. The calculated skin friction coefficient** shows that the model predicts the transition onset to late. On the other hand the transition length is underpredicted, so the position of transition end is captured at right position.

4. Conclusion

The results indicate that the model of Walters and Cokljat (2008) is able (after all necessary corrections with respect to the original article) to predict the laminar-turbulent transition for simple flows over flat plate. Future work will be oriented to the implementation of the model to our in-house code and to its applications for flows in turbine cascades.

**The c_f as well as the Reynolds number Re_x was related to local velocity magnitude $U_{0.05}(x)$ at $y = 0.05$ m.

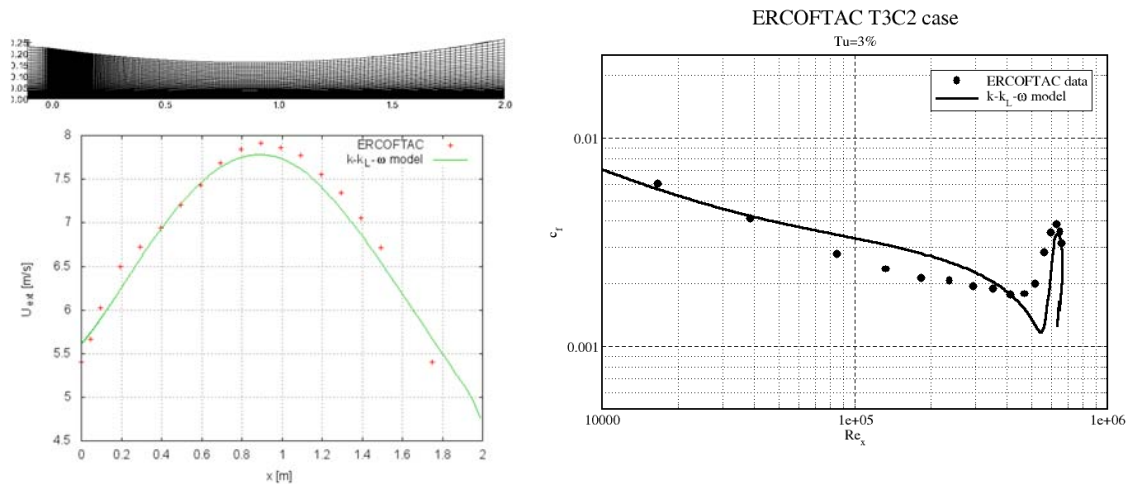


Fig. 2: Domain, the velocity at $y = 0.05\text{m}$ and the friction coefficient for T3C2 ERCOFTAC cases.

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