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MATERIAL NON-LINEAR BEAM ELEMENT WITH SHEAR CAPACITY

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Abstract: *The paper describes a new formulation of beam elements for deformational variant of FEM, which respects non-linear material behaviour. This formulation considers combination of shear and axial loading and is suitable for short or torque beams similarly to the Timoshenko theory. Non-uniform warping and influence of transversal contraction are not considered in the formulation. The cross section can be of arbitrary known shape and composed of more materials. The presented element respect real distribution shear stress over cross section taking material non-linearity into consideration*

Keywords: ***beam element, FEM, material non-linearity, shear, torque***

1. Introduction

More economical usage of materials at technical objects is connected with the development of simulation tools. A similar situation is at beam construction which is abundantly represented in technical practice. Just use of material non-linearity provides reserves in material usage. Besides that in practice more and more emphasis is placed on robustness of simulation tools without neccessarity of deep knowledge of service. Frequent use of some materials makes correct use of linear models impossible, for example reinforced concrete. For these reasons, based on practical requests of users, the following formulation of beam element was developed. Geometry of the presented element is shown on fig. 1.

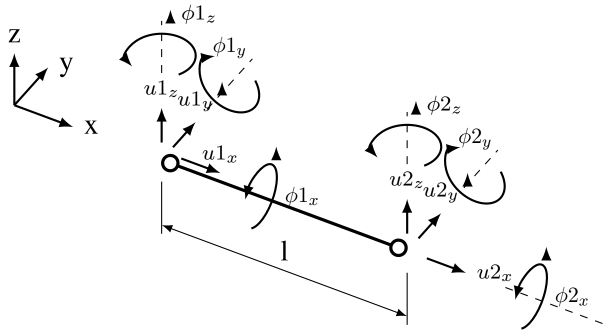

Fig. 1: Beam element

During making solution about used formulation of the element that should be implemented an extensive research of available formulations in commercial CAE software's (Nastran, Abaqus, Marc, Ansys) was done. All softwares provides a lot of formulations of the beam element including material non-linearity. Their definition it is possible to find for example here Crisfield (2000),Němec (2010) and Zienkiewicz (2000). Detail assessment shows that there is provided robust and sufficient solution for Euler formulation of the element for uniaxial stress in all software's. But in case of shear stress the situation is dismal. Here are provided solutions for Timoshenko formulation as well but detail analysis proves that these solutions are very simplified. For example Ansys provides the element BEAM 188/189. This element however supposes constant shear strain over cross section and strain corresponding to linear torsion. But real stress distribution at shear loading is different (for example Grutmann (1999)). In other

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CAE software's the situation is similar. Even literature search did not provide more complex formulation for material non-linearity.

None of available formulations of the beam element respect real distribution shear stress over cross section taking material non-linearity into consideration.

Solution of plastic torque is known for long time (for example here Chakrabarty (2006)). Numeric solution of plastic torque for an arbitrary cross section is defined here Gruttmann (2001). By small modification the author extend this solution to general shear loading Kabeláč (2011). The model for axial part of stress is generally known (good description in Crisfield (2000)). By combining of these solutions we obtain real stress distribution over cross section at arbitrary combination of beam loading. In that way formulated cross section behaviour is combined with appropriate shape function. Result of this combination is here presented formulation of beam element, which respects real shear stress distribution over arbitrary cross section.

Use of various cross section characteristics, specified by an user, increases demands on an user and is a source of possible mistakes. Therefore we left numerical values as input and the only input is a shape of cross section or FEM mesh of cross section. On this mesh more materials can be defined. It means the model is also usable for composite cross section.

The represented model is valid for small strain, straight prismatic beam and free warping of cross section. The influence of transversal contraction is not considered. In connection to co-rotational formulation its involving in geometric non-linearity is easy.

2. Formulation

The final formulation of beam element with the shear influence and considering of material non-linearity is connection of following analyses:

1. **Distribution of deformations along beam**. A classical formulation of beam elements does not seem to be suitable for this propose. A simplified formulation was used with constant strain along axis of beam. From here transversal deformations are interpolated by quadratic function and the other deformations are interpolated linearly.

$$\mathbf{\Psi} = [\frac{du_x}{dx}, \frac{d\phi_y}{dx}, \frac{d\phi_z}{dx}, \frac{d\phi_x}{dx}, \theta_y, \theta_z]^T \quad ; \quad \mathbf{\Psi} = \mathbf{\Xi}.\mathbf{u} \tag{1}$$

2. **Strain over cross section**. In view of the formulation strain over cross section is defined by three components ε_x , γ_{xy} and γ_{xz} the other components are zero. These components are defined by current unknown warping function w and components of strain along axis of beam.

$$\boldsymbol{\varepsilon}_i = \mathbf{B_i}.\mathbf{w_i} + \mathbf{G_i}.\mathbf{\Psi} \tag{2}$$

3. **Calculation of warping function**. Based on described process of strain over cross section and using of equilibrium equations a particular PDE problem can be formulated. Considering used material model it is a non-linear problem. This problem can be by variational principles transformed to classical non-linear FEM problem over mesh of cross section. The result is an actual warping function and then stress over cross section.

$${}^{k}\mathbf{F} = \sum_{i}^{elem} \int_{\Omega_i} \mathbf{B}_i^T.\boldsymbol{\sigma}({}^{k}\boldsymbol{\varepsilon}_i).d\Omega_i = 0 \Rightarrow \mathbf{w} \tag{3}$$

4. **Calculation of cross section internal forces V and tangential stiffness matrix of cross section D**. Based on known warping function and stress over cross section it is easy to define final internal forces and stiffness matrix of cross section.

$$\mathbf{V} = [N_x, M_y, M_z, M_x, V_y, V_z]^T = \sum_{i}^{elem} \int_{\Omega_i} \mathbf{G_i}^T . \boldsymbol{\sigma_i}(\boldsymbol{\varepsilon_i}) . d\Omega_i \quad ; \quad \mathbf{D} = \frac{\partial V_i}{\partial \Psi_j} \tag{4}$$

5. **Internal forces R and tangential stiffness matrix of beam element K** .

$$\mathbf{R} = l.\Xi^T.\mathbf{V} \quad ; \quad \mathbf{K_t} = l.\Xi^T.\mathbf{D}.\Xi \tag{5}$$

2.1. Beam shape function

It is a known and generally widespread solution for Timoshenko beam in linear area. But it is not possible to find a direct relation between deformation in grid and skew from shear by this solution. For this reason a simplified formulation was developed. Process of rotation along beam is linear interpolation of rotation in grid. Shape functions are used here.

$$\eta \in < -1, 1 > \tag{6a}$$

$$N_1 = \frac{1}{2}(1 - \eta) \tag{6b}$$

$$N_2 = \frac{1}{2}(1 + \eta) \tag{6c}$$

$$x = l.N_2 \tag{6d}$$

$$\phi_z = N_1.\phi 1_z + N_2.\phi 2_z \tag{7}$$

It means the element has constant curvature in deformation. In this case bend must be interpolated by quadratic polynomial.

$$u_y = N_1.u1_y + N_2.u2_y + \alpha.Nc \tag{8}$$

$$Nc = 1 - \eta^2 \tag{9}$$

Coefficient α is excluded as follows. Skew is obtained from the relation:

$$\theta_z = -\phi_z + \frac{du_y}{dx} \tag{10}$$

To avoid shear locking of the element, the skew must be constant.

$$\frac{d\theta_z}{dx} = 0 \tag{11}$$

By solution of this equation it is excluded parameter α and for skew and transversal deformation is valid:

$$u_y = N_1.u1_y + N_2.u2_y - \frac{L}{8}(\phi 2_z - \phi 1_z).Nc \tag{12}$$

$$\theta_z = \frac{1}{L}(u2_y - u1_y) - \frac{1}{2}(\phi 1_z + \phi 2_z) \tag{13}$$

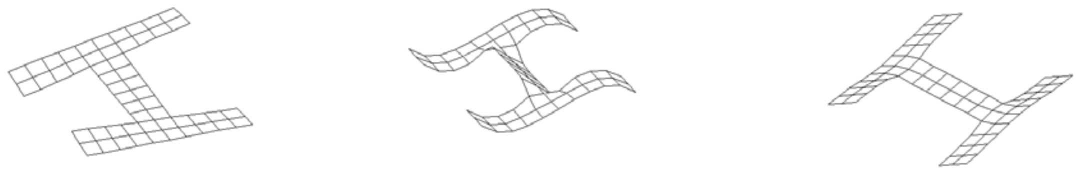
It remains to express the relation between deformation in grid and strain of beam by transforming into 3D:

$$\boldsymbol{\Psi} = \begin{bmatrix} \frac{du_x}{dx} \\ \frac{d\phi_y}{dx} \\ \frac{d\phi_z}{dx} \\ \frac{d\phi_x}{dx} \\ \theta_y \\ \theta_z \end{bmatrix} = \frac{1}{l} \begin{bmatrix} u_{2x} - u_{1x} \\ \phi_{2y} - \phi_{1y} \\ \phi_{2z} - \phi_{1z} \\ \phi_{2x} - \phi_{1x} \\ -u_{2z} + u_{1z} - \frac{l}{2}\left(\phi_{2y} + \phi_{1y}\right) \\ u_{2y} - u_{1y} - \frac{l}{2}\left(\phi_{2z} + \phi_{1z}\right) \end{bmatrix} = \boldsymbol{\Xi}.\mathbf{u} \tag{14}$$

In that way formulated element has a great advantage which lies in a fact that strain of beam are constant.

2.2. Strain over cross section

In accordance with St'Venant theory free warping is occurring over cross section. Free warping is defined by currently unknown warping function $\mathbf{w}$. Examples of warping functions are shown on fig. 2 for different shear loading.



torque shear shear

Fig. 2: Example of warping function on I - profile

For strain over cross section it could be written.

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \gamma_{xy} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\ \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{du_x}{dx} + z\frac{d\phi_y}{dx} - y\frac{d\phi_z}{dx} \\ \frac{\partial w}{\partial y} + \theta_z - z\frac{d\phi_x}{dx} \\ \frac{\partial w}{\partial z} - \theta_y + y\frac{d\phi_x}{dx} \end{bmatrix} \tag{15}$$

Interpolation of warping function $\mathbf{w}$ by shape functions over mesh of cross section pursuant to FEM principles is used. Warping function is clearly defined by values in grids of mesh $\mathbf{w_i}$. If $\mathbf{B}$ is derivation matrix of shape function, than for strain it is possible to write:

$$\varepsilon = \mathbf{B}.\mathbf{w} + \mathbf{G}.\boldsymbol{\Psi} \tag{16}$$

$$\mathbf{G} = \begin{bmatrix} 1 & z & -y & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & 0 & 1 \\ 0 & 0 & 0 & y & -1 & 0 \end{bmatrix} \tag{17}$$

2.3. Derive warping function

In the paper Kabeláč (2011) there was described analysis how to define warping function $\mathbf{w}$ by FEM :

$$\Omega \quad : \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{18a}$$

$$\Gamma \quad : \quad \tau_{xy}.n_y + \tau_{xz}.n_z = 0 \tag{18b}$$

In view of the formulation of the element the first derivation is zero. By using of variational principle and FEM principles the problem is transformed into system of non-linear equations.

$$\mathbf{F} = \sum_{i}^{elem} \int_{\Omega_i} \mathbf{B}_i^T.\boldsymbol{\sigma}(\varepsilon_i).d\Omega_i = 0 \Rightarrow \mathbf{w} \tag{19}$$

$$\mathbf{K_F} = \frac{\partial F_i}{\partial w_j} \tag{20}$$

These equations are supplemented by conditions for warping function.

$$N = \int_{\Omega} w.E.d\Omega = 0 \tag{21a}$$

$$M_y = \int_{\Omega} z.w.E.d\Omega = 0 \tag{21b}$$

$$M_z = \int_{\Omega} -y.w.E.d\Omega = 0 \tag{21c}$$

Which can be written down as:

$$\mathbf{L}.\mathbf{w} = 0 \tag{22}$$

Here arbitrary material model can be used and defined as follows:

$$\boldsymbol{\sigma} = f(\boldsymbol{\varepsilon}) \tag{23}$$

$$\mathbf{D}_m = \frac{\partial f_i}{\partial \varepsilon_j} \tag{24}$$

The result is warping function defined by values in grid of mesh $\mathbf{w_i}$ for actual deformations $\boldsymbol{\Psi}$. More details in Kabeláč (2011).

2.4. Internal forces and tangential stiffness matrix of cross section

To express searching warping function $\mathbf{w}$ it is possible to express internal forces of cross section by simple relation.

$$\mathbf{V} = \sum_{i}^{elem} \int_{\Omega_i} \mathbf{G_i}^T.\boldsymbol{\sigma_i}(\boldsymbol{\varepsilon_i}).d\Omega_i \tag{25}$$

Where $\boldsymbol{\varepsilon_i}$ is expressed by relation (16). It remains to express tangential stiffness matrix of cross section $\mathbf{D}$, whih is obtained by derivation of relation (25) according to $\boldsymbol{\Psi}$.

$$\mathbf{D} = \frac{\partial V_i}{\partial \Psi_j} = \sum_{i}^{elem} \int_{\Omega_i} \mathbf{G_i}^T.\mathbf{D_m}.\left(\mathbf{G_i} + \mathbf{B_i}.\mathbf{dw_i}\right).d\Omega_i \tag{26}$$

$$\mathbf{dw} = \frac{\partial w_i}{\partial \Psi_j} \tag{27}$$

Derivation $\mathbf{w}$ according to $\boldsymbol{\Psi}$ is a result of solution of system of linear equations.

$$\begin{bmatrix} \mathbf{K_F} & \mathbf{L} \\ \mathbf{L}^T & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{dw} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{H} \\ \mathbf{0} \end{bmatrix} \tag{28}$$

$$\mathbf{H} = \sum_i^{elem} \int_{\Omega_i} \mathbf{B_i}^T.\mathbf{D_m}.\mathbf{G_i}.d\Omega_i \tag{29}$$

2.5. Internal forces and tangential stiffness matrix of beam

One last step is necessary to make. Because strain $\boldsymbol{\Psi}$ is constant along axis of beam for internal forces in grids of beam element is valid.

$$\boxed{\mathbf{R} = \int_0^l \mathbf{\Xi}^T.\mathbf{V}.dx = l.\mathbf{\Xi}^T.\mathbf{V}} \tag{30}$$

And for tangential stiffness matrix of beam:

$$\boxed{\mathbf{K} = \int_0^l \mathbf{\Xi}^T.\mathbf{D}.\mathbf{\Xi}.dx = l.\mathbf{\Xi}^T.\mathbf{D}.\mathbf{\Xi}} \tag{31}$$

3. Conclusions

The element presented here is robust enough for practical use. Cross section can have arbitrary origin point. It can be composed of more materials and the formulation is independent on material model. A disadvantage of the element is neglecting of non-uniform warping especially in prevailing shear loading. The element will be tested in future and it is supposed to be implemented in commercial CAE software.

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