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SEQUENTIAL DESIGNS OF EXPERIMENTS FOR SAMPLING-BASED SENSITIVITY ANALYSIS

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Abstract: *The sensitivity analysis is a basic tool for investigating the sensitivity of a model response to its inputs. One widely used strategy to assess the sensitivity is based on a set of simulations for a given sets of input parameters, i.e. points in the design space. An estimate of the sensitivity can be then obtained by computing correlations between the input parameters and the chosen components of the model response. The accuracy of the sensitivity prediction depends on the choice and the number of design points called the design of experiments. Moreover, once the design of experiments is created, the obtained sensitivity prediction may be inaccurate because of the insufficient number of design points. To improve the prediction, new design points should be sequentially added into the existing design. The aim of the presented paper is to review and compare available criteria determining the quality of the sequential design of experiments suitable for sampling-based sensitivity analysis.*

Keywords: ***Sequential design of experiments, Space-filling, Orthogonality, Latin Hypercube Sampling, Sampling-based sensitivity analysis***

1. Introduction

Sensitivity analysis (SA) is an important tool for investigating properties of complex systems. It is an essential part of inverse analysis procedures (Kučerová (2007)) and is also closely related to response surface modelling (Helton et al (2006)) or uncertainty analysis (Helton et al (2006)). To be more specific, SA provides some information about the contributions of individual system parameters/model inputs to the system response/model outputs. A number of approaches to SA has been developed, see e.g. Saltelli et al (2000) for an extensive review. The presented contribution is focused on widely used sampling-based approaches (Helton et al (2006)), particularly aimed at an evaluation of Spearman's rank correlation coefficient (SRCC), which is able to reveal a nonlinear monotonic relationship between the inputs and the corresponding outputs.

When computing the SA in a case of some real system using expensive experimental measurements or some computationally exhaustive numerical model, the number of samples to be performed within some reasonable time is rather limited. Randomly chosen sets of input parameters do not ensure appropriate estimation of related sensitivities. Therefore the sets must be chosen carefully. A review and comparison of several criteria, which can govern the stratified generation of input sets called as a design of experiments (DOE), is presented in Janouchová and Kučerová (2011).

Another important aspect of a DOE generation is a choice of the number of design points. A small DOE does not have to give us the required accuracy of the sensitivity prediction and one has to increase the number of design points so as to achieve the accuracy improvement. Once having the time-consuming measurements for the original small design, adding new points into the existing design is more efficient than generation of the whole larger DOE. The subject of adding new points to the initial design is considered e.g. in Crombecq et al (2011), where the authors use sequential space-filling designs for surrogate modeling. In this paper we follow the results presented in Janouchová and Kučerová (2011) for small DOEs and focus on sequentially generated DOEs based on the chosen criteria and their comparison in terms of sensitivity prediction.

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The following section reviews the criteria for optimization of a DOE, which are available in literature. Section 3. includes some comments on widely used methods for stratified generation of a DOE and Section 4. deals with generation of optimal sequential DOEs. Section 6. presents the comparison of the optimal sequential designs quality for usage in sampling-based SA for theoretical analytical functions, while the results for structural models are given in Section 7. Concluding remarks are summarized in Section 8.

2. Criteria for assessing optimal designs

A number of different criteria for assessing the quality of particular DOE can be found in literature. In general, they can be organized into groups with respect to the preferred DOE property. The most widely preferred features are

- *space-filling* property, which is needed in order to allow for evaluation of sensitivities valid for the whole given domain of admissible input values, so-called design space;
- *orthogonality*, which is necessary to assess the impact of individual input parameters.

Other main objectives can be preferable in particular applications of DOE. In response surface methodology, reduction of noise and bias error can become more important than the orthogonality (Goel et al (2008)). Nevertheless, no special objectives were formulated for the case of sampling-based SA, so we employ the common ones.

2.1. Space-filling criteria

Let us recall four widely used space-filling criteria.

Audze-Eglais objective function (AE) proposed in Audze and Eglais (1977) is based on a potential energy among the design points. The points are distributed as uniformly as possible when the potential energy E^{AE} proportional to the inverse of the squared distances among points is minimized, i.e.

$$E^{\mathrm{AE}} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{1}{L_{ij}^2}\,, \tag{1}$$

where n is the number of the design points and L_{ij} is the Euclidean distance between points i and j.

Euclidean maximin (EMM) distance is probably the best-known space-filling measure (Johnson et al (1990); Morris and Mitchell (1995)). It states that the minimal distance $L_{\min,ij}$ between any two points i and j should be maximal. In order to apply the minimization procedure to all presented criteria, we minimize the negative value of a minimal distance E^{EMM}, i.e.

$$E^{\mathrm{EMM}} = -\min\{..., L_{ij}, ...\}, \quad i = 1...n, \quad j = (i+1)\,...\,n\,. \tag{2}$$

Modified L_2 discrepancy (ML_2) is a computationally cheaper variant of a discrepancy measure, which is widely used to assess precision for multivariate quadrature rules (Fang and Wang (1994)). Here, the designs are normalized in each dimension to the interval $[0, 1]$ and then, the value of ML_2 is computed according to

$$E^{\mathrm{ML}_2} = \left(\frac{4}{3}\right)^k - \frac{2^{(1-k)}}{n} \sum_{d=1}^{n} \prod_{i=1}^{k} (3 - x_{di}^2) + \frac{1}{n^2} \sum_{d=1}^{n} \sum_{j=1}^{n} \prod_{i=1}^{k} [2 - \max(x_{di}, x_{ji})]\,, \tag{3}$$

where k is the number of input parameters, i.e. the dimension of the design space and x_{di} and x_{ji} are the i-th coordinates of the d-th and j-th points, respectively. Since the evaluation of discrepancy for

a large design can be time-consuming, some efficient algorithms are proposed e.g. in Heinrich (1996). To achieve the best space-filling property of DOE, the value of ML_2 should be minimized.

D-optimality criterion (Dopt) was proposed in Smith (1918) as a pioneering work in the field of DOE for regression analysis. This criterion minimizes the variance associated with estimates of regression model coefficients by minimizing the determinant of the so-called dispersion matrix $(\mathbf{Z^T Z})^{-1}$ or equivalently, by maximizing the determinant of the so-called information matrix $(\mathbf{Z^T Z})$ (de Aguiar et al (1995)). Again, in order to apply a minimization procedure, but to avoid the inversion of the information matrix, we can minimize negative value of the determinant of the information matrix, i.e.

$$E^{\mathrm{Dopt}} = -\det(\mathbf{Z^T Z})\,, \tag{4}$$

where $\mathbf{Z}$ is a matrix with evaluated regression terms in the design points. An important shortcoming of this criterion concerns a choice of the number of regression terms to be included into the information matrix, because few terms can lead to the D-optimal DOE with duplicated points. In this contribution we employ a Bayesian approach as described in Hofwing and Strömberg (2010) or Janouchová and Kučerová (2011).

2.2. Orthogonality-based criteria

There are two well-know approaches to evaluate the orthogonality of a DOE. The most popular one is based on correlation among the samples' coordinates, the other one is a conditional number.

Conditional number (CN) is commonly used in numerical linear algebra to examine the sensitivities of a linear system (Cioppa and Lucas (2007)). Here, we use conditional number of $\mathbf{X}^T\mathbf{X}$, where $\mathbf{X}$ is a matrix of the design points' coordinates, so-called design matrix

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}, \tag{6}$$

where n is the number of the design points and k is the dimension of the design space and the columns are centred to sum to 0 and scaled to the range $[-1, 1]$. The conditional number is then defined as

$$E^{\mathrm{CN}} = \mathrm{cond}(\mathbf{X^T X}) = \frac{\lambda_{\mathbf{1}}}{\lambda_{\mathbf{n}}}\,, \tag{7}$$

where λ_1 and λ_n are the largest and smallest eigenvalues of $\mathbf{X}^T\mathbf{X}$, respectively, therefore the E^{CN} is greater or equal to 1. Values closer to 1 correspond to more orthogonal DOE, therefore the conditional number should be minimized.

Pearson product-moment correlation coefficient (PMCC) is a standard measure of a linear dependence between two variables. Having two variables x_i and x_j, the PMCC is defined as

$$c_{ij} = \frac{\mathrm{Cov}\,(x_i, x_j)}{\sigma_{x_i}\sigma_{x_j}} = \frac{\sum_{a=1}^{n}(x_{ai} - \overline{x_i})(x_{aj} - \overline{x_j})}{\sqrt{\sum_{a=1}^{n}(x_{ai} - \overline{x_i})^2 \sum_{a=1}^{n}(x_{aj} - \overline{x_j})^2}}\,, \tag{8}$$

where

$$\overline{x_i} = \frac{1}{n}\sum_{a=1}^{n} x_{ai} \quad \text{and} \quad \overline{x_j} = \frac{1}{n}\sum_{a=1}^{n} x_{aj}\,. \tag{9}$$

The PMCC takes a value between -1 and 1 and positive values indicate that the value of x_i tends to increase together with increasing value of x_j, while negative values indicate decreasing value of x_i with increasing value of x_j. Zero value stands for no linear relationship between x_i and x_j. In order to obtain

orthogonal DOE in a multi-dimensional design space, the PMCC needs to be evaluated for each pair of columns in the design matrix (7). As a result, one obtains a $k \times k$ symmetrical correlation matrix

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kk} \end{bmatrix} . \tag{10}$$

In the case of an orthogonal DOE, the correlation matrix $\mathbf{C}$ is equal to identity matrix. To achieve an orthogonal DOE, one can, for instance, minimize the maximum $|c_{ij}|$ as in Cioppa and Lucas (2007) or the sum of squares of the elements above the main diagonal of $\mathbf{C}$ as it is done in engineering softwares (Novák (2011); Novák et al (2011)) as well as in presented results, i.e.

$$E^{\mathrm{PMCC}} = \sqrt{\sum_{i=1}^{k} \sum_{j=i+1}^{k} c_{ij}^2} \, . \tag{11}$$

Spearman's rank correlation coefficient (SRCC) can be used to capture a nonlinear but monotonic relationship between two variables and therefore, it can be efficiently applied for estimation of correlations in sampling-based SA (Helton et al (2006)). The idea is to replace the values of x_{ai} and x_{aj} by their corresponding ranks $r(x_{ai})$ and $r(x_{aj})$ and then the SRCC can be computed as

$$\rho_{ij} = 1 - \frac{6 \sum_{a=1}^{n} \left(r(x_{ai}) - r(x_{aj}) \right)^2}{n(n^2 - 1)} \, . \tag{12}$$

In case of a multi-dimensional design space, the orthogonality of the DOE can be similarly to (11) achieved by minimizing

$$E^{\mathrm{SRCC}} = \sqrt{\sum_{i=1}^{k} \sum_{j=i+1}^{k} \rho_{ij}^2} \, . \tag{13}$$

Kendall tau rank correlation coefficient (KRCC) is an alternative measure of a nonlinear dependence between two variables. In particular, it is based on the number of concordant ($T_{c,ij}$) and discordant ($T_{d,ij}$) pairs of samples according to

$$\tau_{ij} = \frac{T_{c,ij} - T_{d,ij}}{n(n-1)/2} \, , \tag{14}$$

and again, the orthogonal DOE can be obtained by minimizing

$$E^{\mathrm{KRCC}} = \sqrt{\sum_{i=1}^{k} \sum_{j=i+1}^{k} \tau_{ij}^2} \, . \tag{15}$$

3. Latin Hypercube Sampling

Since the optimization of DOE defined on real domains becomes computationally exhaustive even at moderate number of dimensions or design points, practical applications are usually restricted to the optimization of the so-called Latin Hypercube (LH) designs (Iman and Conover (1980)). LH sampling provides a possibility to represent prescribed probabilistic distribution of particular variables and hence, it can be efficiently applied in uncertainty analysis (Helton et al (2006)). The idea is to divide the range of each variable x_j into n disjoint intervals of equal probability and one value x_{ij} is selected from each interval. This selection can be either random or commonly prescribed to the centre of the interval. Then, the n values for each variable are randomly coupled without replacement with n values of other variables resulting in n vectors of variables where each discrete value of each variable is used only once.

The discretisation itself is quite useful for simplification of the optimization process. Therefore, we focus our attention mainly to the optimization of DOE in discrete domains assuming that continuous domains are usually also discretised so as to make the optimization process manageable. Of course, the LH restriction simplifies the optimization even more, since the search space is significantly reduced. However, it is not obvious, whether such restriction excludes the best solutions regarding the objective of SA. Hence, we examine here both the LH designs as well as *free* designs without any prescribed restrictions on points position.

4. Sequential designs of experiments

When generating experimental designs sequentially, lack of knowledge about the resulting number of design points leads obviously to worse properties of the resulting designs than the one-shot DOEs. Nevertheless, an optimization of a large one-shot DOE can become too complex and thus a sequential design can achieve better qualities than the result of one-shot DOE optimization stuck in a local extreme.

The authors in Crombecq et al (2011) present a review of a different procedures for one by one generated sequential designs and compare their qualities with one-shot DOEs in terms of projected and intersite distance. In general, the aim of the described procedures is to provide DOEs with good space-filling and orthogonal property, but they cannot be used for reliability analysis, since the DOEs do not fulfil any probability distribution.

Other authors focus on LH designs which allow to maintain the probability distributions of particular variables. Vořechovský (2009) proposed strategies for adding new points into an existing LH design based on sequential refinement of the domain discretisation. The methods, however, suffer from highly increasing number of points to be added.

In this contribution we focus on a humble goal to compare the qualities of sequential designs obtained by sequential addition of a constant number of new points, preserving the original discretisation and optimized to the individual criteria described in Section 2.. We assume that adding of a constant number of points, which can be handled by an optimization process, should provide a DOE with better qualities than sequential one by one DOEs.

5. Generation of Optimal Design of experiments

We start our comparison by generating small designs having 10 points in two-dimensional discrete domain. The both variables are defined always in ten equally distant discrete values. Each design is optimized according to one of the criteria described in Section 2.. One set of LH designs is optimized with the LH constrains, the other set of *free* designs is optimized without any constrains.

Since the designs are not excessively complex, the Simulated Annealing method (Kirkpatrick et al (1983); Černý (1985)) was applied to optimize each criterion. The procedure slightly differs for the free and for the LH designs. While in the first case a single loop of the algorithm involves a sequential selection of a design point and its random movement to any unoccupied position, in the latter case the algorithm randomly chooses two points and then switches one of their randomly chosen coordinates. The acceptance of a new solution is driven by the Metropolis criterion

$$\exp\left(\frac{f_{\text{old}} - f_{\text{new}}}{T}\right) \geq U\,, \tag{16}$$

where f_{old} and f_{new} stand for values of a criterion for an actual and for a new solution, respectively. T denotes the algorithmic temperature initially set to $T_{\max} = 10^{-3}$ and gradually reduced by a multiplicative constant $T_{\text{mlt}} = \left(T_{\max}/10^{-6}\right)^{1/100}$ after each $n_{\max}/10$ iterations or sooner if the number of accepted movements reaches the value $n_{\max}/100$. The entire algorithm terminates after $n_{\max} = 10^6$ criterion evaluations.

Of course, there is no guarantee that the global optimum is achieved, nevertheless, more frequent falls to local extremes also reflect the shortcoming of a particular criterion. Hence, we decided to present the obtained results without any deeper search for more robust and reliable optimization method.

The optimization process was performed 100 times for each criterion and the obtained designs were then employed as initial designs for sequential designs generation. Larger designs were generated sequentially in three steps by adding 10 new points into the existing designs. The following optimization process is very similar to the optimization of the initial designs. The difference consists of optimization only of the added points positions, while the optimized criterion is evaluated for the whole new design.

There are presented two methods for generating sequential designs in this paper. The first one is based on unrestricted selection of new points according to the optimized criterion and both the free DOEs as well as LH DOEs were employed as initial ones. Figs. 1 and 2 show examples of resulting designs of this first method. The plotted examples are chosen among the other designs for their worst result in the sum of the minimal distances to other points. The aim is to show the worst results one can obtain by optimizing particular criteria.

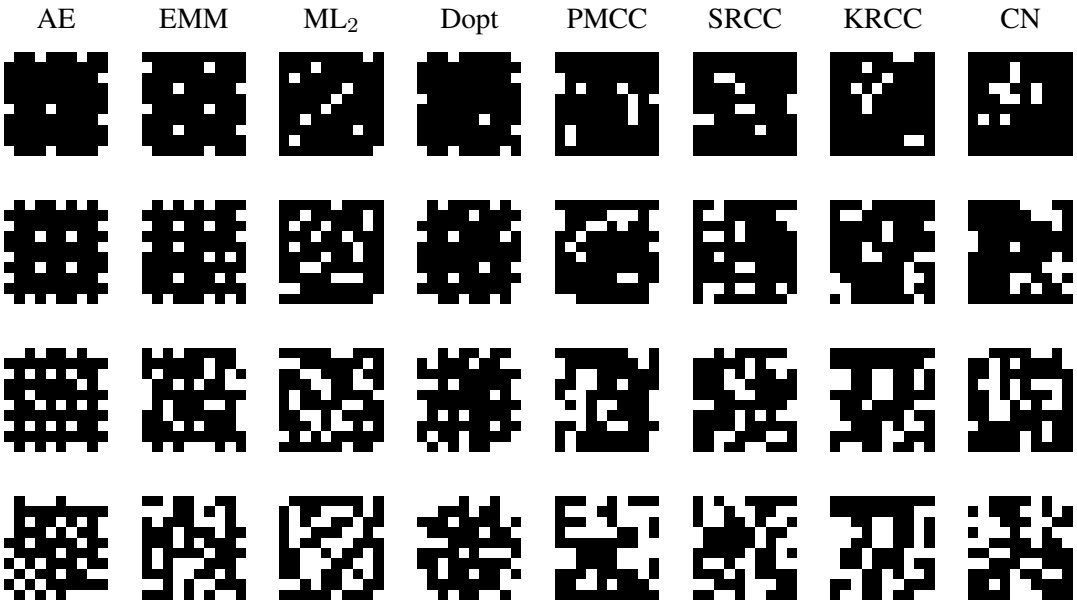


Fig. 1: Sequential free designs.

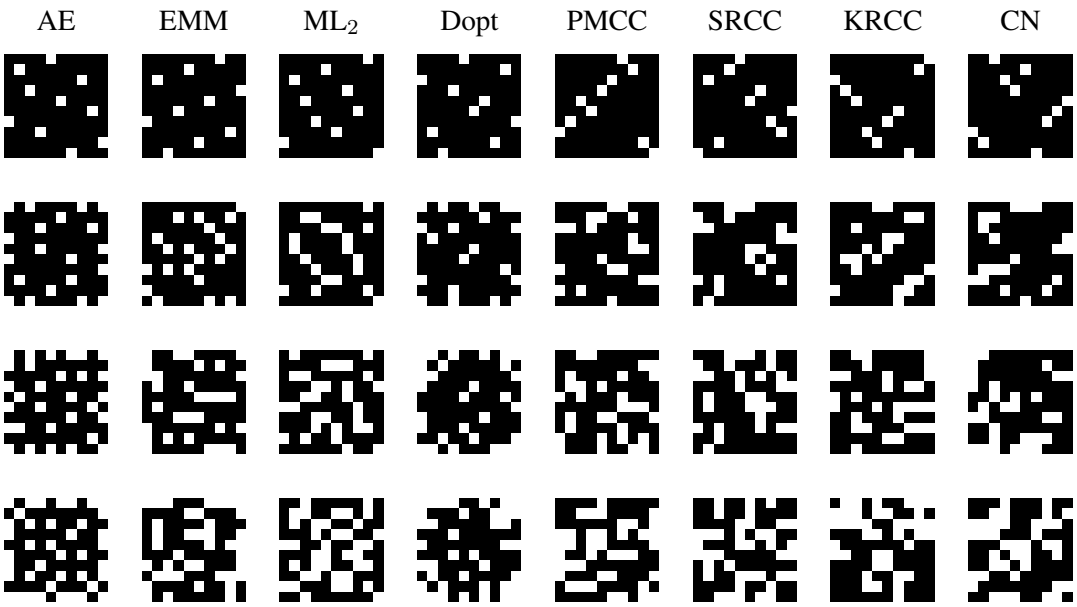


Fig. 2: Sequential free designs from original LH designs.

The second method preserve the LH constrains also for the added points and thus, the equal number of points are located in each column or row. Here the LH designs are used as the initial designs. The resulting DOEs obtained by this method are presented in Fig. 3.

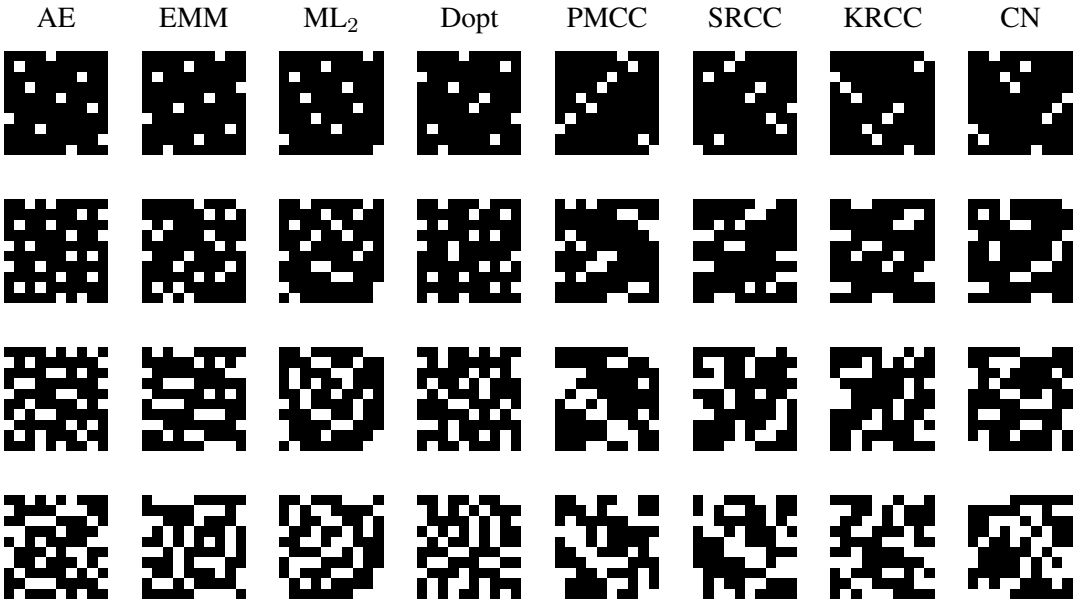


Fig. 3: Sequential mixed LH designs from original LH designs.

6. Sensitivity analysis on a set of mathematical functions

The SRCC criterion has been shown that it can significantly improve the results in estimating the importance of model parameters in sensitivity analysis for the case of nonlinear monotonic models (Helton et al (2006)).

Having the numerical model given as

$$z = f(x_1, x_2, \ldots, x_k) \tag{17}$$

relating the model response z and the model parameters x_i, the impact of the parameter x_i to the model response z can be estimated by evaluating their Spearman's rank correlation $\rho_{x_i,z}$ according to

$$\rho_{x_i,z} = 1 - \frac{6 \sum_{a=1}^{n} \left(r(x_{ai}) - r(z_a)\right)^2}{n(n^2-1)}\,, \tag{18}$$

where x_{ai} are values of particular model parameter corresponding to points in DOE and z_a are values of model responses corresponding to these points.

In engineering practice, the majority of the numerical models fulfil the condition of a monotonic relationship between the model parameters and the model response. Therefore, to support the study of optimal DOE quality in sampling-based SA, we performed a comparison for a list of nonlinear but monotonic models. The shapes of the chosen models plotted for the case of square 10×10 domain are shown in Figure 4 together with corresponding parameter-response correlations obtained for the **Full** design consisting of all feasible design points (here 100 points).

Then, the parameter-response correlations were estimated using the all obtained optimal designs and the differences among correlations $\tilde{\rho}$ obtained by the optimal designs and correlations ρ obtained by the full designs are stored. The error measure ϵ in the parameter-response correlations evaluated for a given function is considered as an average difference between each parameter and model response correlation obtained by an optimal and a full design, i.e.

$$\epsilon = \frac{1}{k} \sum_{i=1}^{k} \left| \tilde{\rho}_{x_i,z} - \rho_{x_i,z} \right|. \tag{19}$$

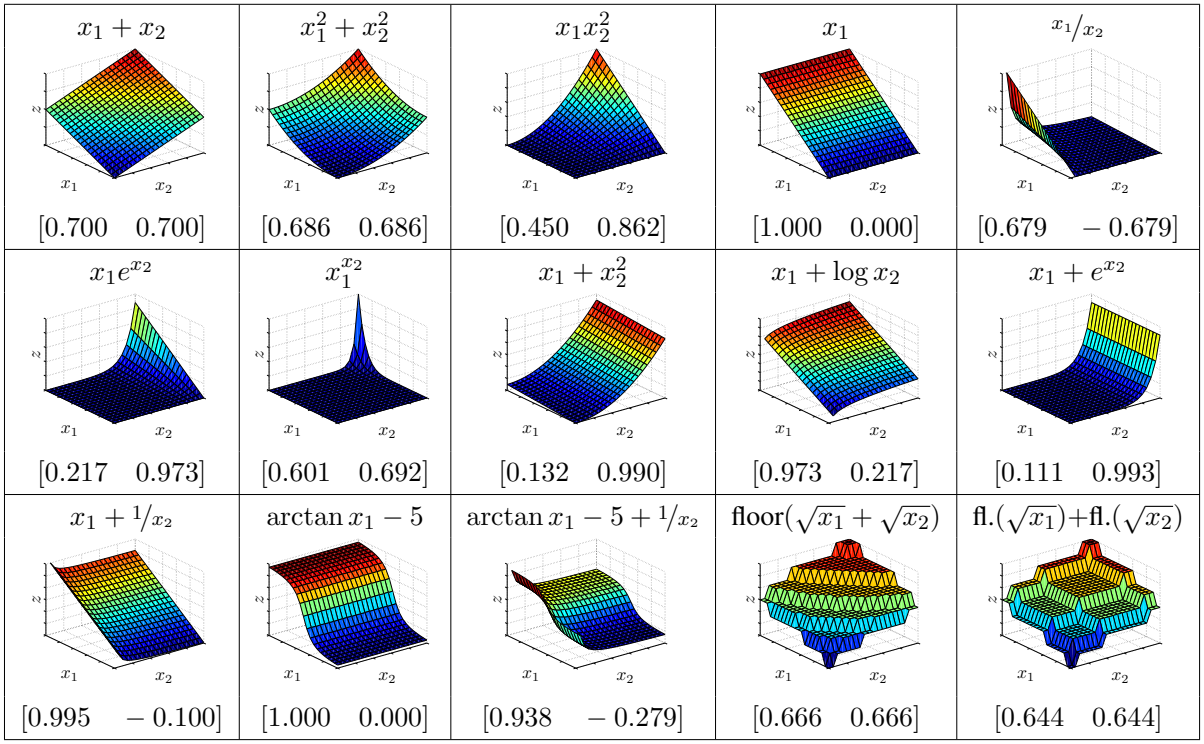


Fig. 4: Shapes of 15 functions for sensitivity analysis with the corresponding values of parameter-response correlations.

The statistics over the obtained values of errors ϵ is presented in Figure 5 using the box plots.

For an easier evaluation of particular criteria, the mean and maximal errors over all models multiplied by 100 are listed in Table 1.

Tab. 1: Mean and maximal errors in correlation predictions for theoretical models.

			AE		**EMM**		**ML$_2$**		**Dopt**		**PMCC**		**SRCC**		**KRCC**		**CN**	
		points	mean	max	mean	max	mean	max	mean	max	mean	max	mean	max	mean	max	mean	max
1st method	free	10	5.9	32.0	6.2	29.4	5.1	20.5	4.5	33.6	8.4	51.4	7.3	48.5	7.5	46.5	8.6	59.9
		20	3.3	16.5	4.6	19.7	3.1	14.3	2.9	20.7	5.1	34.2	4.4	32.4	4.4	29.9	4.4	37.4
		30	2.1	12.0	4.3	17.1	2.2	10.6	3.0	17.4	3.6	25.2	3.3	22.7	3.2	26.1	2.9	22.0
		40	1.5	10.0	4.1	16.0	1.7	8.5	2.5	13.0	2.6	19.0	2.4	23.4	2.5	19.6	2.2	17.0
		overall	**3.2**	**17.6**	**4.8**	**20.6**	**3.0**	**13.5**	**3.2**	**21.2**	**4.9**	**32.5**	**4.4**	**31.8**	**4.4**	**30.5**	**4.5**	**34.1**
	LH	10	6.7	28.9	9.8	31.1	4.5	11.2	6.8	20.3	5.5	31.4	5.8	28.4	5.6	33.0	5.3	30.7
		20	2.9	20.2	5.6	23.8	2.8	13.5	3.5	20.3	4.2	25.2	3.9	31.4	3.9	25.7	3.6	25.7
		30	2.0	12.6	4.5	16.5	2.0	9.5	3.0	15.3	3.1	19.1	2.9	23.7	2.9	20.0	2.7	22.6
		40	1.8	10.6	4.3	18.6	1.7	8.3	2.4	13.6	2.5	16.5	2.3	16.9	2.3	14.8	2.2	17.9
		overall	**3.4**	**18.1**	**6.1**	**22.5**	**2.8**	**10.6**	**3.9**	**17.4**	**3.8**	**23.1**	**3.7**	**25.1**	**3.7**	**23.4**	**3.5**	**24.2**
2nd method	LH	10	6.7	28.9	9.8	31.1	4.5	11.2	6.8	20.3	5.5	31.4	5.8	28.4	5.6	33.0	5.3	30.7
		20	4.8	15.4	7.1	25.3	2.8	10.2	6.0	19.3	3.1	20.8	3.1	18.5	3.0	18.6	3.0	25.2
		30	2.6	11.3	5.3	22.9	2.3	8.7	2.5	9.6	2.1	21.8	2.2	18.2	2.3	17.5	2.2	20.5
		40	2.5	9.0	4.9	19.2	1.6	5.9	2.3	8.8	1.6	11.3	1.6	12.6	1.8	15.5	1.7	15.1
		overall	**4.2**	**16.2**	**6.8**	**24.6**	**2.8**	**9.0**	**4.4**	**14.5**	**3.1**	**21.3**	**3.2**	**19.4**	**3.2**	**21.2**	**3.1**	**22.9**

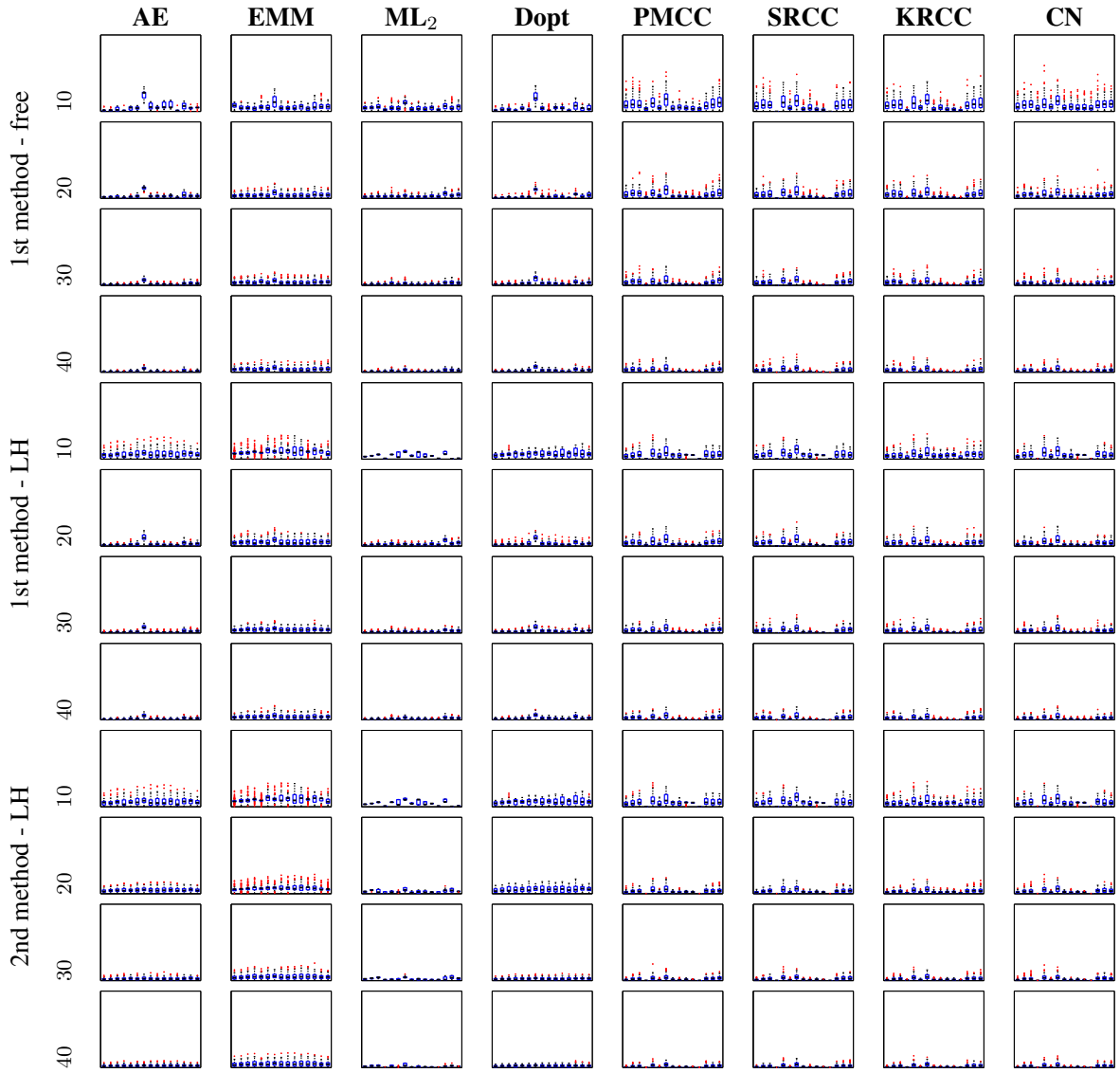
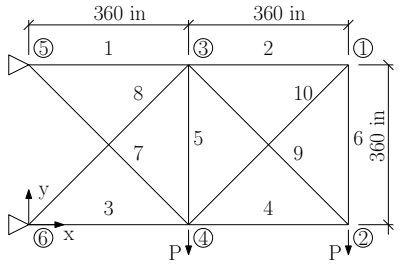


Fig. 5: Statistics on results of criteria in estimating parameter-response correlations for theoretical models. Each rectangle contains 15 *box plots representing the distribution of errors in correlation prediction for particular models depicted in Figure 4. The scale is* 0 *at the bottom and* 1 *at the top per rectangle. Furthermore, each rectangle refers to the optimal designs according to one criterion associated with the corresponding column and to the chosen number of the design points and the type of restriction applied associated with the corresponding row.*

7. Sensitivity analysis on truss structures

The sensitivity analysis study presented in the previous section is aimed on two-dimensional theoretical problems. Therefore, this section is devoted to illustrative engineering problems with higher number of dimensions. We have chosen two models of truss structures commonly used as benchmarks for sizing optimization.

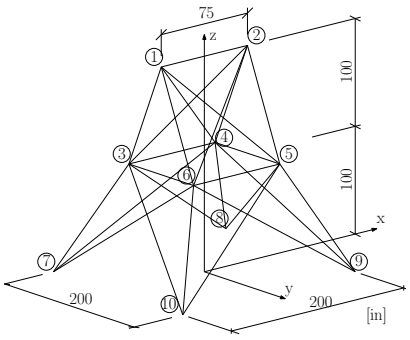
The first one represents a ten-bar truss structure (Venkaya (1971)) shown in Figure 6. The design variables are the cross-sectional areas of the bars. This benchmark is defined with two types of variables: continuous and discrete one. Here we focus on a discrete formulation with discrete values of cross-sectional areas together with values of material properties and loading taken from Lemonge and Barbosa (2003). All ten cross-sectional areas have the same 42 feasible discrete values (in^2): 1.62, 1.80, 1.99, 2.13, 2.38, 2.62, 2.63, 2.88, 2.83, 3.09, 3.13, 3.38, 3.47, 3.55, 3.63, 3.84, 3.87, 3.88, 4.18, 4.22, 4.49, 4.59, 4.80, 4.97, 5.12, 5.74, 7.22, 7.97, 11.50, 13.50, 13.90, 14.20, 15.50, 16.00, 16.90, 18.80, 19.90, 22.00, 22.90, 26.50, 30.00, 33.50.



Material:	Aluminum
Specific weight:	0.1 lb/in^3
Young's modulus:	10^7 psi
Loading P:	100 kips

Fig. 6: Scheme of a ten-bar truss structure together with material parameters and values of applied loading.

The second model concerns a 25-bar truss structure with a geometry, material properties and loading given in Figure 7. Thanks to the symmetry of the structure, the bars can be organized into eight groups. The bars in one group have the same cross-sectional areas and hence, there are only eight design variables. These variables are again discrete with 30 feasible values (in^2): 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.8, 3.0, 3.2, 3.4, see **?**.



Material:	Aluminum
Specific weight:	0,1 lb/in^3
Young's modulus:	10^7 psi

Nodal loads			
Node	F_x	F_y	F_z
1	1.0	-10.0	-10.0
2	0.0	-10.0	-10.0
3	0.5	0.0	0.0
6	0.6	0.0	0.0

Fig. 7: Scheme of a 25-bar truss structure together with material parameters and values of applied loading.

The response of these models consists of three components: total weight of the structure w, maximal deflection d and maximal stress s. Because of higher dimensions of these problems, we have generated the optimal DOEs only with LH restriction, since they can be optimized more easily. This restriction automatically specifies the number of corresponding design points, which has to be equal to the number of feasible values (42 for the ten-bar truss and 30 for the 25-bar truss).

A simulated annealing method with the same parameters as described in Section 5. but with a maximum of 10^7 iterations was employed to generate 20 optimal DOEs using each criterion under the study. The parameter-response correlations were then estimated using the obtained optimal DOEs and compared with the correlations computed using the **Large** DOEs consisting of $2 \cdot 10^7$ samples generated by Monte Carlo method. The statistics on the errors ϵ in estimation of parameter-response correlations is demonstrated again in terms of box plots independently for each criterion and each model response component, see Figure 8.

For clearer summarization of the obtained results, the mean and maximal errors in correlation predictions multiplied again by 100 are listed in Table 2.

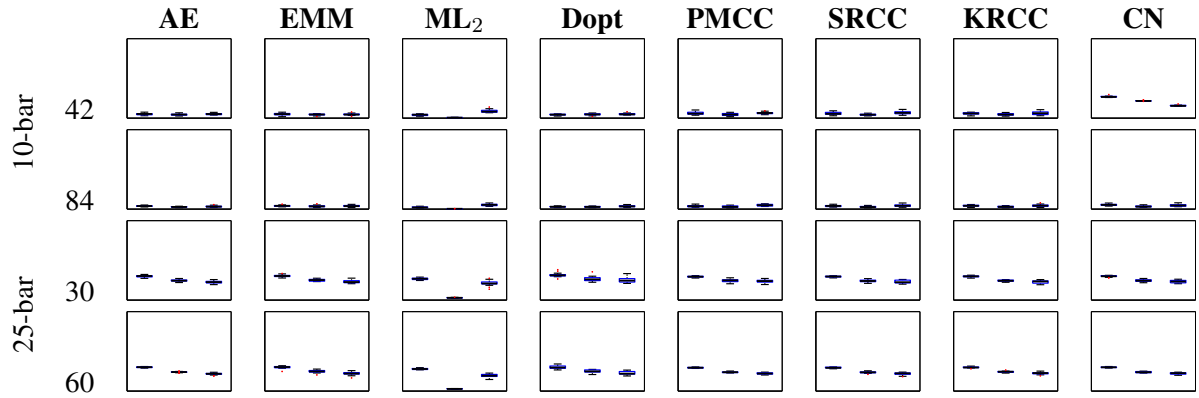


Fig. 8: Statistics on results in estimation of parameter-response correlations for structural models. Each rectangle contains three box plots representing the distribution of the errors in correlation prediction for particular model responses: the total weight w, the maximal deflection d and the maximal stress s. The scale is 0 at the bottom and 1 at the top per rectangle. Each rectangle refers to optimal designs according to one criterion associated with the corresponding column and to the chosen structure associated with the corresponding row.

Tab. 2: Mean and maximal errors in correlation predictions for structural models.

Model		AE		EMM		ML_2		Dopt		PMCC		SRCC		KRCC		CN	
		mean	max	mean	max	mean	max	mean	max	mean	max	mean	max	mean	max	mean	max
10-bar	42 w	5.1	7.5	4.9	7.5	4.1	5.5	4.3	5.4	6.2	10.3	5.9	9.2	5.8	7.5	27.0	29.7
	42 d	4.5	6.7	4.4	5.6	0.7	1.1	4.6	6.5	4.5	7.3	4.3	6.2	4.7	7.1	21.7	23.0
	42 s	5.1	7.3	4.7	7.7	8.9	13.9	5.1	8.0	6.5	9.2	6.8	11.0	6.6	10.7	15.7	17.7
	42 overall	**4.9**	**7.5**	**4.7**	**7.7**	**4.6**	**13.9**	**4.7**	**8.0**	**5.7**	**10.3**	**5.7**	**11.0**	**5.7**	**10.7**	**21.5**	**29.7**
	84 w	3.9	5.1	4.1	6.1	2.3	3.6	3.0	4.2	3.7	6.3	3.9	6.1	4.1	5.8	5.4	7.7
	84 d	2.7	3.6	3.6	6.7	0.4	0.7	2.9	4.0	3.1	5.0	2.7	4.6	2.9	4.4	3.4	5.5
	84 s	3.3	5.7	3.9	6.0	5.4	7.7	3.7	5.8	5.0	6.7	4.3	7.7	4.4	8.0	4.8	8.0
	84 overall	**3.3**	**5.7**	**3.9**	**6.7**	**2.7**	**7.7**	**3.2**	**5.8**	**3.9**	**6.7**	**3.6**	**7.7**	**3.8**	**8.0**	**4.5**	**8.0**
25-bar	30 w	30.0	32.3	30.4	33.5	26.8	28.9	31.6	38.3	29.5	30.8	29.6	30.7	29.9	31.4	30.0	31.2
	30 d	24.6	27.2	25.1	27.5	23.6	28.3	26.8	35.7	24.4	28.1	24.1	27.1	24.3	25.8	24.5	27.2
	30 s	22.6	25.9	23.2	27.6	21.2	27.4	25.2	33.4	23.8	27.2	23.0	26.0	22.7	25.8	23.4	26.3
	30 overall	**25.7**	**32.3**	**26.2**	**33.5**	**23.9**	**28.9**	**27.9**	**38.3**	**25.9**	**30.8**	**25.6**	**30.7**	**25.6**	**31.4**	**26.0**	**31.2**
	60 w	30.0	30.9	30.0	31.9	27.8	29.5	30.2	34.1	29.4	30.4	29.4	30.4	29.9	31.4	29.9	30.9
	60 d	23.9	25.1	24.7	27.7	22.5	25.5	24.9	27.3	23.8	25.3	23.6	25.6	24.2	26.5	23.8	25.1
	60 s	21.6	23.7	22.3	25.8	19.3	22.6	22.6	26.4	22.1	23.9	21.9	23.9	22.5	25.1	22.2	23.8
	60 overall	**25.2**	**30.9**	**25.7**	**31.9**	**23.2**	**29.5**	**25.9**	**34.1**	**25.1**	**30.4**	**25.0**	**30.4**	**25.5**	**31.4**	**25.3**	**30.9**

8. Conclusions

This paper reviews eight criteria used for sequential optimization of a design of experiments and presents a comparison of the resulting designs when employed for a sampling-based SA on 15 theoretical and two structural models. The overall results can be summarized in several following conclusions:

- The quality of all the optimal DOEs can be improved by adding of new points and the conclusions valid for small designs remains mostly valid also for the larger sequential designs.
- The best result was achieved by the ML_2 criterion, which is very robust and thus the obtained DOEs provided very small errors in sensitivity predictions with very small variance. The LH designs optimized with respect to ML_2 criterion provide better results and they are also more easily optimized. Therefore they can be recommended for the practical usage.
- Orthogonality-based criteria provide good results on theoretical problems when applied for generation LH designs, but the resulting errors have higher variance and which lower their fidelity.

- An interesting effect can be noticed on D-optimal designs. While the small D-optimal free designs have significantly better results than the small D-optimal LH designs, this difference decreases with increasing number of points and the larger D-optimal LH designs have the same or smaller errors than the D-optimal free designs of the same size.
- Also AE criterion achieved very good results when applied to free designs, but the variance is again relatively high.
- Worse results were obtained using EMM criterion.

The presented results revealed that the ML_2 criterion, which is not very popular, can provide very good results when applied for the generation of the DOE for sampling-based sensitivity analysis. The results obtained for 25-bar structure were unsatisfactory for all the studied criteria. This can be caused by too small number of samples in the generated designs. However, the larger designs did not improve the results significantly.

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