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IMPLEMENTATION OF DIRECT NUMERICAL SIMULATION OF VISCOUS INCOMPRESSIBLE FLOW

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Abstract: *The method for solving viscous incompressible flows, based on Glowinski, Pan,Hesla & Joseph (1998), is presented for the direct numerical simulation of viscous incompressible flow.It uses a finite element discretization in space and an operator-splitting technique for discretization in time. Quadratic approximation is employed for velocity flow and linear approximation for pressure on triangle elements. The goal is to develop more efficient and accurate numerical tool for computing viscous flows. The accuracy of the presented method has been confirmed on two common cases by implementation in Matlab program: the Poiseuille flow test, and on driven cavity flow test.*

Keywords: ***finite element, liquid flow, operator splitting, Navier-Stokes equations.***

1. Introduction

The aim of this article is to present a method for solving viscous incompressible flows. The finite element implementation is based on the incompressible Navier-Stokes equations on structured two dimensional triangular meshes using operator splitting for time discretization (Glowinski 2003). The fractional-time-step scheme described by Marchuk (1990) has been employed. Liquid is supposed to be incompressible and Newtonian. The advantage of this method is that no repeated remeshing is required. Also the assembled mass matrix remains constant and so it does not have to be assembled at every time step. By splitting one time step into three successive substeps the discretization enables better approximation of results. Pressure is computed in this case in the first sub-step while velocity is computed at each substep. To discretize the domain the Taylor-Hood triangular elements have bed used with second degree polynomic approximation of velocity and first degree polynomic approximation of pressure.

2. Problem formulation

2.1. The governing equations

Assume incompressible viscous Newtonian fluid occupying at the given time $t \in (0, T)$, delimited domain $\Omega \subset \mathbf{R}^2$ with boundary Γ. Let denote by $x = \{x_i\}_{i=1}^2$ a generic point in Ω. Let further denote by $u(x,t)$ velocity and by $p(x,t)$ pressure, both governed by Navier-Stokes equations:

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \rho \mathbf{g} + \nabla.\boldsymbol{\sigma} \text{ on } \Omega \text{ - momentum equation,} \tag{1}$$

$$\nabla.\boldsymbol{u} = 0 \text{ on } \Omega. \text{ – incompressibility condition,} \tag{2}$$

where ρ is density of the fluid, $\mathbf{u}$ is velocity of the fluid, and $\boldsymbol{\sigma}$ is fluid stress. For an incompressible Newtonian viscous fluid, the stress is decomposed into its hydrostatic and shear components,

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta\mathbf{D}[\mathbf{u}], \tag{3}$$

where p is hydrostatic pressure in the fluid, η is the viscosity (assumed constant), and $2\eta\mathbf{D}[\mathbf{u}]$ is rate-of-strain tensor.

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Relations (1)-(3) are to be supplemented by the appropriate initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0 \text{ on } \Omega, \tag{4}$$

$$\nabla.\mathbf{u}_0 = 0, \tag{5}$$

and the boundary conditions

$$\mathbf{u} = \mathbf{u}_\Gamma(t) \text{ on } \Gamma, \tag{6}$$

$$\int_\Gamma \mathbf{u}_\Gamma(t)\,\hat{\mathbf{n}} = 0, \tag{7}$$

where $\hat{\mathbf{n}}$ is unit normal vector pointing out of the Γ.

2.2. Finite Element formulation

Let introduce spaces of approximation and test functions of velocities and pressure:

$$\mathbb{W}_h = \{v_h \in C^0(\Omega)^2, \forall T \in \mathcal{T}_h\},$$

$$\mathbb{W}_{oh} = \{v_h \in \mathbb{W}_h, v_h = 0 \; on \; \Gamma\},$$

$$L_h = \{q_h \in C^0(\Omega), \forall T \in \mathcal{T}_h\},$$

$$L_{0h} = \left\{q_h \in L_h \middle| \int_\Omega q_h dx = 0\right\}.$$

By using these finite-dimensional spaces we arrive at the following finite-element approximation of the problem (1)-(7):

Find $\mathbf{u}_h \in \mathbb{W}_{0h}$, $p \in L_{0h}$ satisfying:

$$\int_\Omega \left(\rho\left(\frac{\mathrm{d}\mathbf{u}_h}{\mathrm{d}t}\right) + (\mathbf{u}_h.\nabla)\mathbf{u}_h\right).\mathbf{v}_h dx - \int_\Omega p_h \nabla.\mathbf{v}_h dx + \int_\Omega 2\eta\mathbf{D}[\mathbf{u}] : \mathbf{D}[\mathbf{v}]dx = 0 \tag{8}$$

for all $\mathbf{v}_h \in \mathbb{W}_{0h}$,

$$\int_\Omega q_h \nabla.\mathbf{u}_h = 0 \text{ for all } q_h \in L_h, \tag{9}$$

$$\mathbf{u}_\mathrm{h}(0) = \mathbf{u}_{0h} \text{ on } \Omega, \tag{10}$$

where $\mathbf{u}_{0h}$ is an approximation of $\mathbf{u}_0$ satisfying the compatibility conditions

$$\int_\Omega q_h \nabla.\mathbf{u}_{0h} = 0 \text{ for all } q_h \in L_h. \tag{11}$$

Since, in (8), $\mathbf{u}$ is divergence-free and satisfies a Dirichlet boundary condition on all of Γ, we can write:

$$\int_\Omega 2\eta\mathbf{D}[\mathbf{u_h}] : \mathbf{D}[\mathbf{v_h}]dx = \int_\Omega \eta\nabla\mathbf{u_h} : \nabla\mathbf{v_h}\, dx \text{ for all } \mathbf{v}_h \in \mathbb{W}_{0h}.$$

3. Time discretization by operator splitting

3.1. Principle operator splitting:

As stated by Glowinski (2003) numerical solutions of the relations (1)-(7) is not trivial due to the following reasons:

- The above equations are nonlinear
- Handling of the incompressibility condition (2)
- The above equations are system of partial differential equations, coupled through the nonlinear term $(\boldsymbol{u}\,.\nabla)\boldsymbol{u}$, the incompressibility condition $\nabla.\boldsymbol{u} = 0$, and sometimes through the boundary conditions.

In the following sections, we show that a time discretization by operator splitting will partly overcome the above difficulties; in particular, decoupling of difficulties associated with the nonlinearity from those associated with the incompressibility condition. To introduce operator splitting, the approach by Glowinski & Pironneau (1992) will be used. Let assume the following initial value problem:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} + A(\varphi) = 0,$$

$$\varphi(0) = \varphi_0,$$

where A is an operator (possibly nonlinear, and even multivalued) from a Hilbert space H into itself and where $\varphi_0 \in$ H. There are numerous splitting operators to solve this problem (see Glowinski 2003).

3.2. Fractional-step scheme

In this work, the fractional-step scheme described by Marchuk (1990) is employed. Let assume decomposition of the operator A into the following nontrivial decomposition:

$$A = A_1 + A_2 + A3,$$

(by nontrivial, we mean A1, A2 and A3 are individually simpler than A). In the following, Δt is a time step.

Set $\varphi^0 = \varphi_0$, for n>= 0, φ^n being known we compute $\varphi^{n+1/3}$, $\varphi^{n+2/3}$ and φ^{n+1} as follows

$$\frac{\varphi^{n+\frac{1}{3}} - \varphi^n}{\Delta t} + A_1\left(\varphi^{n+\frac{1}{3}}\right) = f_1^{n+1},$$

$$\frac{\varphi^{n+\frac{2}{3}} - \varphi^{n+\frac{1}{3}}}{\Delta t} + A_2\left(\varphi^{n+\frac{2}{3}}\right) = f_2^{n+1},$$

$$\frac{\varphi^{n+1} - \varphi^{n+\frac{2}{3}}}{\Delta t} + A_3(\varphi^{n+1}) = f_3^{n+1}.$$

By applying operator splitting to the problem (8-11) we obtain (with $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta = 1$):

Find: $\mathbf{u}^{n+1/3} \in \mathbb{W}_h$ and $p^{n+1/3} \in L_h$

$$\rho \int_\Omega \frac{\mathbf{u}^{n+1/3} - \mathbf{u}^n}{\Delta t} . \mathbf{v} dx - \int_\Omega p^{n+1/3} \nabla . \mathbf{v} dx = 0 \text{ for all } \mathbf{v}_h \in \mathbb{W}_{0h} \tag{12}$$

$$\int_\Omega q \nabla . \mathbf{u}^{n+1/3} dx = 0 \text{ for all } q_h \in L_h. \tag{13}$$

Find: $\mathbf{u}^{n+2/3} \in \mathbb{W}_h$

$$\rho \int_\Omega \frac{\mathbf{u}^{n+2/3} - \mathbf{u}^{n+1/3}}{\Delta t} . \mathbf{v} dx - \rho \int_\Omega \left(\mathbf{u}^{n+1/3} . \nabla\right) \mathbf{u}^{n+2/3} . \mathbf{v}\, dx + 2\alpha\eta \int_\Omega \mathbf{D}\left[\mathbf{u}^{n+2/3}\right] : \mathbf{D}[\mathbf{v}]\, dx = 0 \tag{14}$$

for all $\mathbf{v}_h \in \mathbb{W}_{0h}$.

Finally find $\mathbf{u}^{n+1} \in \mathbb{W}_h$

$$\rho \int_\Omega \frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+2/3}}{\Delta t} . \mathbf{v} dx + 2\beta\eta \int_\Omega \mathbf{D}[\mathbf{u}^{n+1}] : \mathbf{D}[\mathbf{v}]\, dx = 0 \text{ for all } \mathbf{v}_h \in \mathbb{W}_{0h}. \tag{15}$$

3.3 Finite element approximation

The triangular Taylor-Hood element has been used, with the quadratic velocity and linear pressure interpolation (see Figure 1).

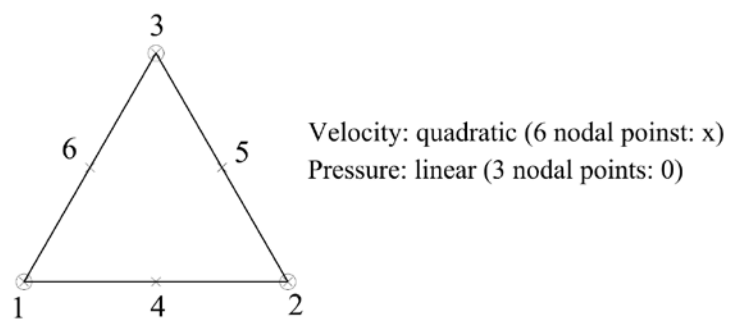

Figure 1: The Taylor-Hood element.

The interpolation functions for velocity have following term:

$$N_1 = 1 - 3\xi - 3\eta + 4\xi\eta + 2\xi^2 + 2\eta^2,$$
$$N_2 = -1\xi + 2\xi^2,$$
$$N_3 = -1\eta + 2\eta^2,$$
$$N_4 = 4\xi - 4\xi\eta - 4\xi^2,$$
$$N_5 = 4\xi\eta,$$
$$N_6 = 4\eta - 4\xi\eta - 4\eta^2,$$

for pressure, linear interpolation is used

$$L_1 = 1 - \xi - \eta,$$
$$L_2 = \xi,$$
$$L_3 = \eta,$$

where ξ, η are isoparametric coordinates.

The particular variables are approximated on the elements as linear combinations of interpolation functions and nodal values:

$$u(\xi,\eta) = \sum_{i=1}^{6} N_i u_i,$$
$$v(\xi,\eta) = \sum_{i=1}^{6} N_i v_i,$$
$$p(\xi,\eta) = \sum_{i=1}^{3} L_i p_i,$$

where u_i, v_i a p_i are the corresponding nodal values. The concept of isoparametric elements is used, where the geometry of an element is approximated using quadratic interpolation:

$$x(\xi,\eta) = \sum_{i=1}^{6} N_i x_i,$$
$$y(\xi,\eta) = \sum_{i=1}^{6} N_i y_i,$$

where $x_i$and $y_i$are the coordinates of the node points in the element. By differentiating this, we express the derivative operators as

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix},$$

The integrals in the weak form are evaluated numerically using the Gaussian quadrature formula (Sarada and Nagaraj 2010).

This method has been implemented in the Matlab; yet further, particles including extensions and modification of liquid to fresh concrete flow will already be implemented in the C++.

4. Numerical validations:

The accuracy of prototype Matlab implementation has been verified using two benchmark problems: the Poiseuille flow between parallel plates and driven cavity flow.

4.1. Poiseuille flow

In this classic test, the steady state velocity and pressure distributions is simulated for a fluid moving laterally between two plates whose length and width is much greater than a distance separating them. The geometry, boundary and initial conditions are illustrated on Fig 2. The domain is divided into 400 elements, the height is 1 and the length is 4. The stationary profile of velocity profile at outflow is quadratic. The mass density is $\rho = 1.0\ kg/m^3$, and the viscosity is $\eta = 10^{-2} Pa\ s$. The results correspond with analytic solution.

When time step $\Delta t = 0.005$ is chosen the implemented simulations renders results in time $t = 1{,}5\ s$

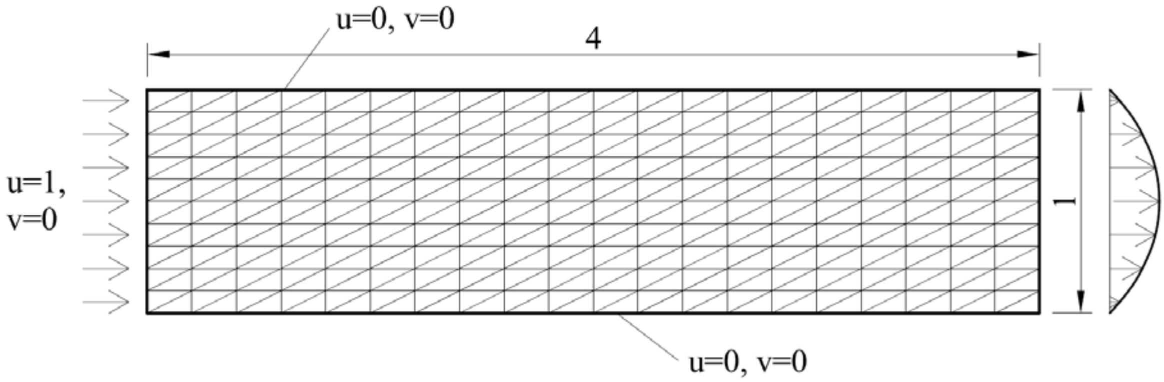

Figure 2: The geometry and the boundary conditions of flow in tube test. The used mesh consists of 861 nodes and 400 elements.

4.2. Driven Cavity flow

The driven cavity flow is a typical problem applied to verify numerical model in fluid dynamics. We simulate the flow inside closed cavity and compare the results with the model developed by the other researches (Botella and Peyret, 1998; Rabenold, 2005). Figure 3 shows geometry of the problem, computational mesh and applied boundary conditions. The viscosity is set to $\eta = 10^{-2} Pa\ s$, and the Reynold's number is computed as $1/\eta$ (based on geometry of size 1 and maximum velocity 1). The mass density is $\rho = 1.0\ kg/m^3$, and the time step is $\Delta t = 0.01$. The results are in good agreement, even though the present values are restricted to the 30x30 grid points (see table No. 1).

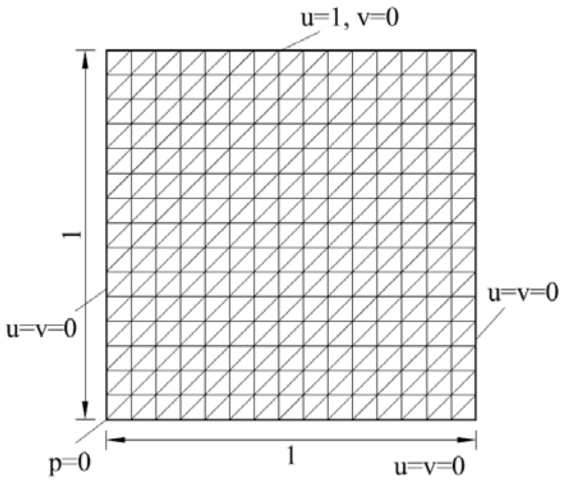

Figure 3: The geometry and the boundary conditions of driven cavity flow test. The used mesh consists of 961 nodes and 450 elements.

Tab. 1: Velocity extreme through cavity centerlines at $Re = 100$

method	u_{min}	v_{min}	v_{max}
Present	-0.2165	-0.2493	0.1771
Botella and Peyret	-0.21279	-0.25266	0.17854
Rabenold	-0.2140424	-0.2538030	0.1795728

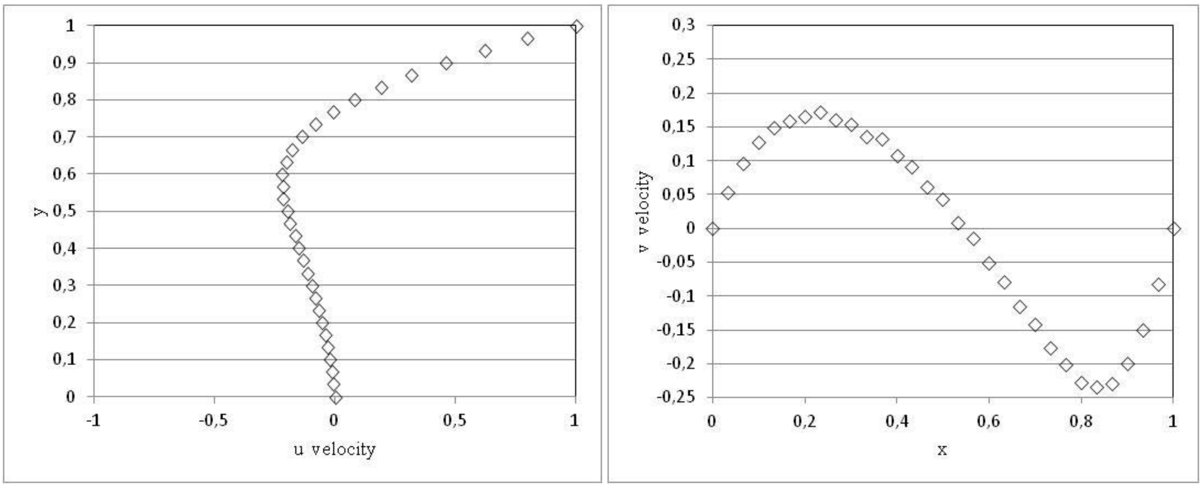

Figure 4: Velocity profiles through cavity centerlines at $\eta = 10^{-2} Pa\ s$ and 30 x 30 grid size.

5. Conclusion

Presented work describes formulation and implementation of non-stationary, incompressible flow finite element solver. The Tayler-Hood elements (P2-P1) have been implemented. For time discretization the operator splitting method is used and it reduces computation of velocity and pressure together to one time substep whereas in other substeps only velocity is being solved. The model is verified using standard benchmark tests from the literature.

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