19th International Conference
ENGINEERING MECHANICS 2013
Svratka, Czech Republic, May 13 – 16, 2013

# NEW MATHEMATICAL MODEL OF CONTINUUM MECHANICS

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***Abstract:*** *This paper presents a variant of a mathematical model of continuum mechanics. Adaptation of the model is focused on unsteady term. The solution is based on the assumption of zero value of the divergence vector, which can have a different physical meaning.*

**Keywords:** ***operator equation, momentum equation, continuum mechanics, Maxwell equations.***

## 1. Introduction

Solution of many problems of continuum mechanics is based on the method of control volumes. Formulation of the task is often complicated by the unsteady term on which in the classic formulation cannot be used Gauss Ostrogradsky theorem.

The above mentioned unsteady term often complicates the problematic of the solid-liquid interaction. Major complications occur even in dealing with the interactions of fields of different physical nature; for example the interaction of the fluid and electromagnetic fields.

The aim of this paper is to modify the mathematical model, so that it was possible to use Gauss Ostrogradsky methods for the unsteady term integration.

The solution comes out a certain type of operator equations, which is typical for a wide class of the continuum mechanics problems.

## 2. Mathematical model

Considering multiple contiguous region *V* bounded by the surface *S*. The boundary orientation is defined by a unit vector outward normal ***n*** to the surface *S*.

The mathematical model is defined by the Cartesian coordinate system, the Euler approach, where each independent variable, generally designated ***E*** depends on the spatial coordinate ***x*** and time ***t***. Thus $\mathrm{E} = \mathrm{E}(\mathbf{x}, \mathrm{t}), \quad \mathbf{x} = (\mathrm{x_j})$. In the *V* there is defined the variable field ***E***.
The mentioned area is defined by a mathematical model using the summation convention in the form:

$$\frac{\partial A_i}{\partial t} + \frac{\partial B_{ij}}{\partial x_j} = C_i \tag{1}$$

$$\frac{\partial A_i}{\partial x_i} = 0 \tag{2}$$

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In the equations (1), (2), we assume:

$$A_i = A_i(\boldsymbol{x}, t)\,; \qquad B_{ij} = B_{ij}[\boldsymbol{A}(\boldsymbol{x}, t)]\,; \qquad A_i = A_i(\boldsymbol{x}, t) \tag{3}$$

$$C_i = C_i(\boldsymbol{x}, t)\,;$$

$$div\boldsymbol{C} = 0 \tag{4}$$

The move to a new model follows; if we apply on the equation (1) the divergence operation and considering (4), we may write:

$$\frac{\partial B_{ij}}{\partial x_i \partial x_j} = 0, \qquad \text{or also} \qquad \frac{\partial B_{jk}}{\partial x_j \partial x_k} x_i = 0. \tag{5}$$

After integrating (5) over the *V* we obtain:

$$\int_S \frac{\partial B_{jk}}{\partial x_k} x_i\, n_j dS = \int_V \frac{\partial B_{jk}}{\partial x_k} \frac{\partial x_i}{\partial x_j} dV \tag{6}$$

Considering $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$, using Gauss Ostrogradsky theorem we obtain:

$$\int_S \left( \frac{\partial B_{jk}}{\partial x_k} x_i - B_{ij} \right) n_j = 0 \tag{7}$$

Expression (7) is very important for the qualitative analysis, it determines the relationship between $\frac{\partial B_{jk}}{\partial x_k}$ and $B_{ij}$ on the boundary S, without the influence of unsteady term $\frac{\partial A_i}{\partial t}$:

Into (7) we substitute now from (1). The following holds:

$$\int_S \left[ \left( \frac{\partial A_j}{\partial t} - C_j \right) x_i + B_{ij} \right] n_j dS = 0 \tag{8}$$

Expression (8) is crucial for solving of the interactions, because the influence of unsteady term is transformed to the boundary *S*, compared to the original model (1), where for the integration over the *V* and the use of Gauss Ostrogradsky theorem applies:

$$\int_V \frac{\partial A_i}{\partial t} dV + \int_S B_{ij}\, n_j dS = \int_V C_i dV \tag{9}$$

Hence it is clear that the analysis and control volume numerical method greatly complicate integrals over the volume *V*.

To a new mathematical model we move simply using the Gauss Ostrogradsky theorem on the expression (8). Hence, it is clear that it applies:

$$\int_V \frac{\partial}{\partial x_j} \left[ \frac{\partial A_j}{\partial t} x_i + B_{ij} - C_j x_i \right] dV = 0 \tag{10}$$

Considering that the term has to pay for each volume *V*, can be placed inside the integral term is zero . So shortly:

$$\frac{\partial}{\partial x_j} \left( \frac{\partial A_j}{\partial t} x_i + B_{ij} - C_j x_i \right) = 0 \tag{11}$$

With regard to (2) also

$$\frac{\partial A_i}{\partial x_i} = 0 \tag{12}$$

Equation (11) , (12) now form a new mathematical model for a class of operators of continuum mechanics.

The model is not only suitable for the method of control volumes, but especially for the qualitative analysis of the interaction forces in the fields of different physical nature; for example for the determination of the solids/liquid interaction, or a magnetic field.

Evidence of the transition from the model (1) to (11) follows from the following identity; let us put:

$$\frac{\partial A_i}{\partial t} = \frac{\partial}{\partial x_j}\left(A_j x_i\right) = \frac{\partial A_j}{\partial x_j} x_i + A_j \frac{\partial x_i}{\partial x_j} \tag{13}$$

Considering the validity of (2) and $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$, the validity of (13) is proved. For the same reason holds $C_i = \frac{\partial}{\partial x_j}\left(C_j x_i\right)$, since we assume (4).

If $C_i$only a function of time, it holds that

$$C_i = C_i(t), \tag{14}$$

Can be used:

$$C_i = \frac{\partial}{\partial x_i}\left(C_j x_j\right) \tag{15}$$

## 3. Class problems of continuum mechanics, which can be transferred to the general shape (1), (2) , respectively (11) , (12).

**A. The equation for the vortex velocity $\mathbf{\Omega} = \mathbf{rot v}$** (Brdicka et al., 2000; Sedov, 1976)

$$\frac{\partial \mathbf{\Omega}}{\partial t} + rot(\mathbf{\Omega} \times \boldsymbol{v}) = 0 \tag{16}$$

$$div\mathbf{\Omega} = 0 \tag{17}$$

Comparing with (1) , (2) holds:

$$\boldsymbol{A} = \mathbf{\Omega}\,; \quad B_{ij} = \varepsilon_{ijk}\varepsilon_{kmn}\Omega_m v_n \tag{18}$$

**B. Maxwell equations: equation for magnetic field strength H assuming infinite conductivity** (LD Landau & EM Lifshitz 2003; Sedov,1976)

$$\frac{\partial \mathbf{H}}{\partial t} + rot(\mathbf{H} \times \boldsymbol{v}) = 0 \tag{19}$$

$$div\boldsymbol{H} = 0$$

Comparing with (1), (2) and (16) , (17), by analogy applies:

$$\boldsymbol{A} = \mathbf{H}\,; \quad B_{ij} = \varepsilon_{ijk}\varepsilon_{kmn}\Omega_m v_n \tag{20}$$

**C. Equations of equilibrium macroscopic particles - Navier Stokes equations** (Brdicka et al., 2000; Sedov,1976)

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j}\left(v_i v_j - \frac{\sigma_{ij}}{\rho}\right) = g_i \tag{21}$$

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{22}$$

Comparing with (1) , (2) holds:

$$A_i = v_i\,; \quad B_{ij} = v_i v_j - \frac{\sigma_{ij}}{\rho}\,; \quad C_i = g_i \tag{23}$$

**D. Homogenous conductor with constant conductivity and permeability**

$$\frac{\partial \mathbf{H}}{\partial t} + rot(\mathbf{H} \times \boldsymbol{v}) - \alpha \Delta \boldsymbol{H} = 0 \tag{24}$$

$$div\boldsymbol{H} = 0 \tag{25}$$

$$\boldsymbol{A} = \mathbf{H}\,; \quad B_{ij} = \varepsilon_{ijk}\varepsilon_{kmn}\mathrm{H}_m v_n - \alpha \frac{\partial H_i}{\partial x_j} \tag{26}$$

When we use our model of (11), it's possible to write operators (16), (19), (21), (24) in the shape:

$$\left[\frac{\partial \mathbf{\Omega}}{\partial t} + rot(\mathbf{\Omega} \times \boldsymbol{v})\right]_i = \frac{\partial}{\partial x_j}\left[\frac{\partial \Omega_j}{\partial t} x_i + \varepsilon_{ijk}\varepsilon_{kmn}\Omega_m v_n\right] = 0 \tag{27}$$

$$\left[\frac{\partial \mathbf{H}}{\partial t} + rot(\mathbf{H} \times \boldsymbol{v})\right]_i = \frac{\partial}{\partial x_j}\left[\frac{\partial H_j}{\partial t} x_i + \varepsilon_{ijk}\varepsilon_{kmn}\mathrm{H}_m v_n\right] = 0 \tag{28}$$

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j}\left(v_i v_j - \frac{\sigma_{ij}}{\rho}\right) - g_i = \frac{\partial}{\partial x_j}\left[\frac{\partial v_j}{\partial t} x_i + v_i v_j - \frac{\sigma_{ij}}{\rho} - g_j x_i\right] = 0 \tag{29}$$

$$\left[\frac{\partial \mathbf{H}}{\partial t} + rot(\mathbf{H} \times \boldsymbol{v}) - \alpha \Delta \boldsymbol{H}\right]_i = \frac{\partial}{\partial x_j}\left[\frac{\partial H_j}{\partial t} x_i + \varepsilon_{ijk}\varepsilon_{kmn}\mathrm{H}_m v_n - \alpha \frac{\partial H_i}{\partial x_j}\right] = 0 \tag{30}$$

## 4. Conclusion

A new mathematical model of a certain class of problems of continuum mechanics was presented. The model can be used for application of the method of finite volumes and qualitative examining of the interactions of the environment with the different physical characteristics. Considering div$\mathbf{A} = 0$ has the original model form:

$$\frac{\partial A_i}{\partial t} + \frac{\partial}{\partial x_j} B_{ij} = C_i \tag{31}$$

Its new, equivalent shape is shown by term:

$$\frac{\partial}{\partial x_j}\left[\left(\frac{\partial A_j}{\partial t} - C_i\right) x_i + B_{ij}\right] = 0 \tag{32}$$

Additionally it holds following identity:

$$\int_V \frac{\partial \boldsymbol{A}}{\partial t} dV = \int_S \left(\frac{\partial \boldsymbol{A}}{\partial t}\boldsymbol{n}\right)\boldsymbol{x} dS \tag{33}$$

**Acknowledgement**

This work was gratefully acknowledged by grant projects: GAČR P101/13-20031S, CZ.1.07/2.3.00/30.0005

**Nomenclature :**

**A, B, C, E** - general dependent variable

$\boldsymbol{x}$ – position vector

$t$ - time

**n -** normal vector

$V$- volume

$S$ - surface

$2\boldsymbol{\Omega}$ - angular velocity

$\boldsymbol{v}$ - velocity vector

$\delta_{ij}$ - Kronecker delta

$\varepsilon_{ijk}$ - Levi-Civita tensor

**References:**

Brdička,M., Samek, L., Sopko, B.: Mechanika kontinua. In Czech Academia, Prague, 2000, ISBN 80-200-0772-5.( in Czech)

LD Landau, EM Lifshitz: Elektrodynamika splošnych Sred. Série "Teoretická fyzika VIII", Moskva, 2003, p 656, ISBN 5-9221-0123-4 (T. VIII) (in Russian)

Sedov, L. I.:Mechanika splošnych sred. TOM 1. Nauka, Moskava, 1976, p.536 (in Russian)

Pochylý, F.; Fialová, S.; Krausová, H. Variants of Navier-Stokes Equations. In Engineering Mechanics 2012; Book of extended abstracts. Svratka, CZ, Association for Engineering Mechanics. 2012. p. 1011 - 1016. ISBN 978-80-86246-40-6.

Pochylý, F.; Fialová, S.; Kozubková, M.; Bojko, M. Tensors of added damping, stiffness and mass in a thin gap of hydrodynamic machine. IOP Conference Series: Earth and Environmental Science. 2012. 15(7). p. 1 - 8. ISSN\~1755-1315.