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MATHEMATICAL DESCRIPTION OF BEHAVIOUR OF A HUMAN BODY DURING WALKING – STANCE PHASE

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Abstract: *The presented paper deals with a mathematical description of behaviour of a human body during walking. Especially the work is aimed at the stance phase of walking process, where the pedestrian's foot is in permanent contact with the ground. If we simplify the human body as a mass point, which is placed in the CoM (center of mass), we are able to state that the pedestrian is acting as an inverted mathematical pendulum, which means that mass of the pendulum hinge is neglected. Thus the vertical force component in the hinge of inverted pendulum corresponds to the time behaviour of the contact force experimentally observed. The simplified pedestrian model was considered as two dimensional, defined in the plane of walking. The solution of this problem was executed in the central gravity field.*

Keywords: Stance Phase of Walking, Inverted Pendulum Model, Runge – Kutta fifth-order method, Ground Reaction Force.

1. Introduction

The human walking is quite complex type of motion, therefore there were introduced some simple models, which allows the mathematical description of this problem by a system of the second-order differential equations. These models of passive walkers are the Rimless wheel model (Margaria, 1976) and the Bipedal walking model (Goswami et. al., 1997). Both of these models (or some of the other models) are based on the behaviour of the mathematical pendulum or inverted mathematical pendulum. This is the reason why the goal of this paper is the numerical study of behaviour of the inverted mathematical pendulum. Respectively it is aimed at the mathematical description of behaviour of a contact force during a stance phase of human gait cycle. The gait cycle is the periodical process, which could be divided into next phases, the stance phase and the swing phase. Detailed description of the human gait cycle was determined by (Vaughan, 1992).

2. Mathematical Modeling

Firstly the equations of motion of the inverted pendulum had to be deduced (see Fig. 1). These equations could be determined by a different approaches e.g. Newton's or D'Alambert approach. Note that the difference between these two assumptions is only in a process of assembling the equations of motion. The Newton's principle uses the principle of equivalence, where the acting force is proportional to the acting acceleration. On the other hand, the D'Alambert's principle takes into account a force of inertia and uses an equilibrium conditions. These procedures are appropriate for the simple or unconstrained dynamical systems. For more complex or constrained systems the Lagrange's analytical approach is more suitable. The equations of motion are derived from the Lagrange-Euler's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, 2, \ldots, n \tag{1}$$

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where L is the Lagrange's functional (Lagrangian), defined by the equation (3), $\dot{q}_i$ and q_i are unknown generalized velocities and generalized spatial coordinates

$$L(t, q_i, \dot{q}_i) = T - U \tag{2}$$

where T is the kinetic energy of the whole system and U is the potential energy. The Lagrangian depends on generalized coordinates and its velocities and implicitly dependent on the time.

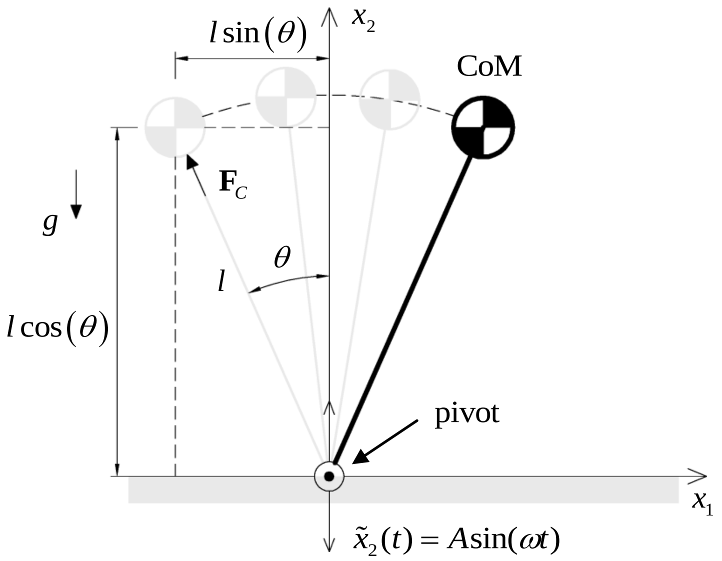

Fig. 1: The scheme of the inverted pendulum model with shaking pivot.

3. Derivation of the Equations of Motion of Inverted Pendulum Model

The well-known simple equation of motion of the inverted 2D pendulum with massless hinge is defined by the equation

$$\ddot{\theta}(t) - g/l \sin(\theta(t)) = 0 \tag{3}$$

where $\ddot{\theta}(t)$ is the angular acceleration, g is the gravitational acceleration, l is the length of the pendulum hinge and $\theta(t)$ is the angle. This second-order ODE could be linearized, with assumption that $\theta(t)$ is small, according to the Taylor-Mc Laurin's Series, thus $\sin\theta(t) \approx \theta(t)$. The system, described by the equation (3), is stabilized by the vertical harmonic motion of the pivot (see Fig. 1). This motion was considered as harmonic function, which is described by the formula

$$\tilde{x}_2(t) = A\sin(\omega t) \tag{4}$$

where A is the constant amplitude, ω is the circular frequency. If the shaking pivot is considered and appropriate time derivatives are executed (see equation (1)), the kinetic energy of the system could be written in the form

$$T = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m \left(\omega^2 A^2 \cos^2(\omega t) - 2\omega A \dot{\theta} l \cos(\omega t) \sin(\theta) \right) \tag{5}$$

And the potential energy of system as

$$U = mgl\cos(\theta) \tag{6}$$

After assembling the Lagrange's functional (2) and executing the appropriate partial and total derivatives (1) the equation of motion of the inverted pendulum stabilized by shaking pivot could be obtained as

$$\ddot{\theta}(t) - \sin(\theta(t)) \left(g/l - \omega^2 A/l \sin(\omega t) \right) = 0 \tag{7}$$

The range of stability was expressed via the Mathieu's equation which is described by relation

$$\frac{d^2\theta}{d\tau^2}+\left[\delta+\varepsilon\cos(\tau)\right]\theta=0 \tag{8}$$

This equation was derived with assumption that θ is small and by using the substitution $\tau=\omega t$, $\delta=-\left(\frac{\omega_0}{\omega}\right)^2$ where $\omega_0^2=g/l$ and $\varepsilon=\frac{A}{l}$. The stability is ensures if $\varepsilon/\sqrt{\delta}<\sqrt{2}$ *(Smith, Blackburn et. al. 1992)*, note that the amplitude $A \ll L$.

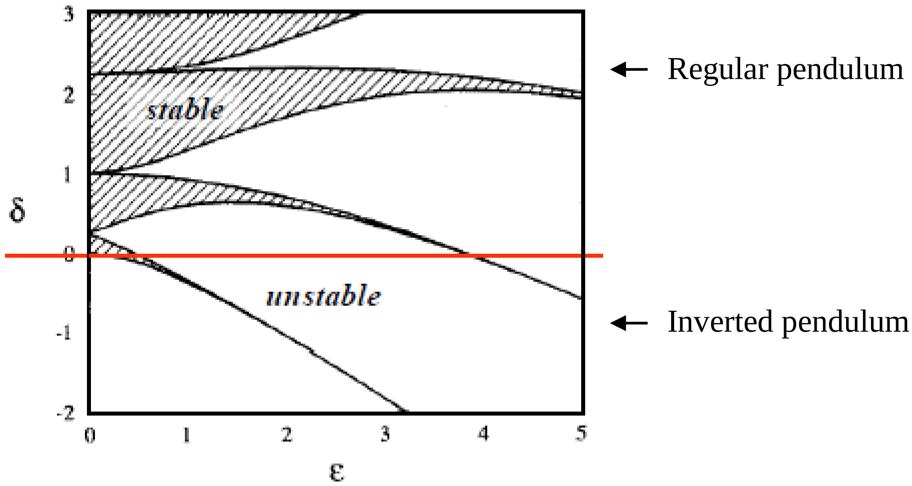

Fig. 2: The region of stability of regular and inverted pendulum (Smith, Blackburn et. al., 1992).

4. Numerical Solution

The numerical solution of the equation (7) was executed via the Runge-Kutta fifth order method with initial conditions $\theta(t)=-\pi/8\ rad$ and $\dot{\theta}(t)=1.8\ rad\cdot s^{-1}$. The relation between the angular and translational velocity is expressed as

$$v(t)=\dot{\theta}(t)l \tag{9}$$

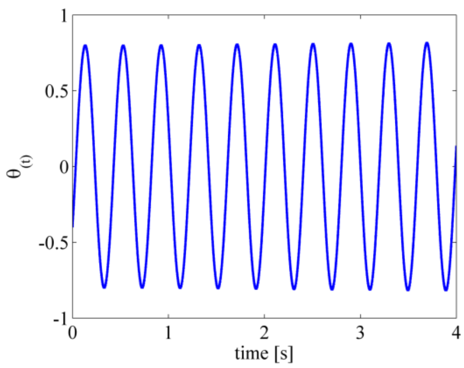

Fig. 3: Time behaviour of the angle $\theta(t)$ of stabilized inverted pendulum.

The vertical component of the ground reaction force (GRF) could be defined simply as

$$F_{x2}^{C}=m\ddot{x}_{2CoM}+mg \tag{10}$$

where $\ddot{x}_{2CoM}$ is the acceleration of the Center of the Mass (CoM) in vertical direction and mg is the force due to gravity. The vertical acceleration of the point mass was transformed into the polar coordinates by differentiation the relation $x_2=l\cos(\theta)$ with respect to the time. Thus the GRF was obtained in the form

$$F_{x2}^{C}=mg-ml\left(\ddot{\theta}\sin(\theta)+\dot{\theta}^2\cos(\theta)\right) \tag{11}$$

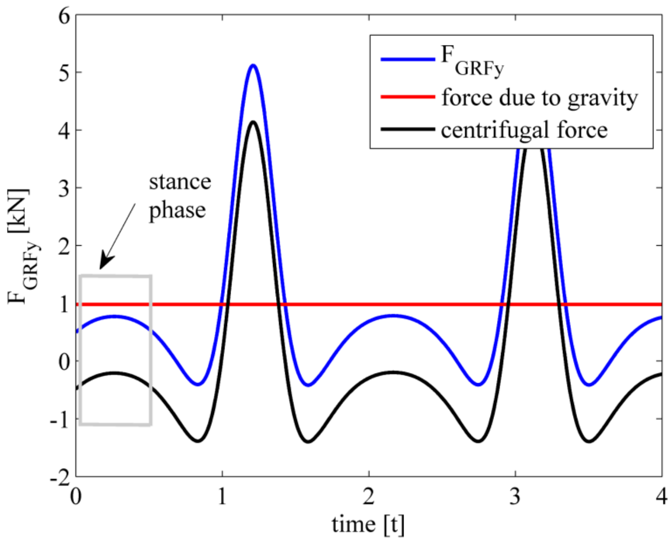

Fig. 4: Time behaviour of the vertical component of the GRF computed on model of inverted pendulum.

5. Conclusion

The mathematical description of the stance phase of human gait cycle was presented in this paper. Human body locomotion should be simplified and described by the movement of the inverted pendulum during a stance phase, which has been stabilized by the vertically vibrating pivot. The obtained time behaviour of the stabilized inverted pendulum is shown in the Fig. 3. As it is obvious from the Fig. 3 a stable periodic motion is obtained due to the stabilization process of the system described by the equation (7).

The numerical study of the inverted mathematical pendulum showed a good accordance with the assumption that the decrease of the contact force during a stance phase of walking occurs. The decrease of the GRF is caused by the transmission of the body weight of a pedestrian. Thus the model of the inverted pendulum provides a good mathematical description of behaviour of the human body during stance phase.

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