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TRANSIENT RESPONSE OF LAYERED ORTHOTROPIC STRIP TO TRANSVERSE LOAD

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Abstract: *This work concerns the transient response of an infinite two-layered strip subjected to a transverse load of impact character. The material of each layer is assumed to be specially orthotropic, i.e. the material and geometric axes coincide. Moreover, the material is modelled as linear viscoelastic using the model of standard linear viscoelastic solid such that the damping behaviour of the strip for long wavelengths and long times can be addressed. The non-stationary wave phenomena in the strip are studied using analytical approach. The system of equations of motion for the case of 2D plane-stress problem is solved using the classical method of integral transform. Once the formulas for the Laplace transforms of fundamental mechanical quantities are derived, the numerical inverse Laplace transform is used to obtain the response in time domain for a strip with free-fixed boundaries. The results for a strip composed of two orthotropic layers of specific material properties are presented in this work. Finally, this solution is confronted with the results of numerical simulations reached by a professional FE code.*

Keywords: Wave propagation, Layered strip, Orthotropic material, Viscoelastic material, Analytical solution.

1. Introduction

Propagation of stationary and transient stress waves through layered structures is the subject of intensive interest for many years. This interest is related to the application of layered materials as vibration isolators and impact absorbers. Most of existing works concerning the optimal design of layered structures is based on numerical approaches but several papers making the use of analytical methods exist, e.g. Luo et al. (2009) or Velo & Gazonas (2003). Analytical and semi-analytical approaches enable deeper insight into the problem and more efficient design process in such cases.

This work deals with the transient wave problem of an infinite viscoelastic strip composed of two orthotropic layers. The analytical solution of the problem with free-fixed boundaries is derived by means of classical method of integral transforms. Fourier and Laplace transforms are applied in spatial and time domains, respectively. This paper follows our previous works Adámek & Valeš (2015) and Adámek et al. (2015) in which the solutions for a single-layer strip problem and for a two-layered strip with free-free boundaries are presented.

2. Problem formulation

The scheme of the problem solved is depicted in Fig. 1. We will assume an infinite strip composed of two layers of the same thickness d and of special orthotropic properties such that the material and geometric axes coincide. Each layer will be identified by the index n, the index n = 1 corresponds to the lower layer, while n = 2 denotes the upper layer. Further, the material of both layers will be assumed to be linear viscoelastic and the model of standard orthotropic viscoelastic solid will be used for its representation (see Sobotka, 1984).

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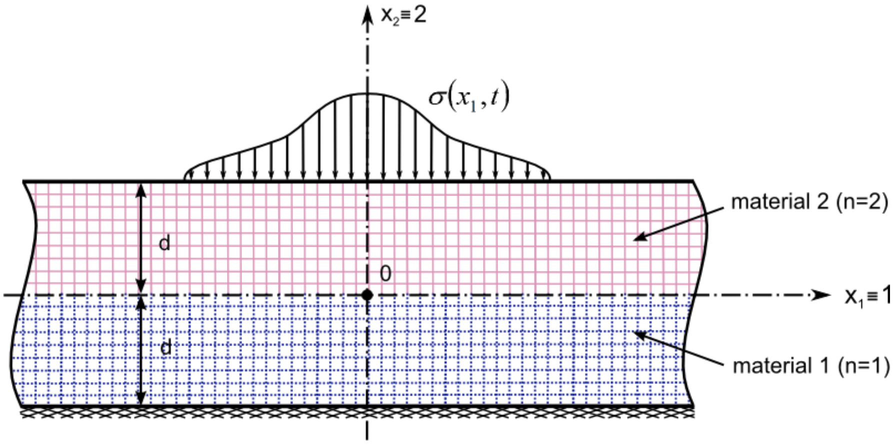

Fig. 1: The scheme of the problem solved.

The boundary and initial conditions of the problem can be specified as follows: (i) the upper edge of the strip is loaded in vertical direction by pressure/tension described by a function even in horizontal coordinate; (ii) the lower edge of the strip is fixed, i.e. zero vertical and horizontal displacements are prescribed; (iii) zero initial conditions for displacement components and their time derivatives will be assumed. Given the above, the problem will be solved as a plane stress problem in the coordinate system x_1-x_2 which is advantageously chosen in such a way that both axes represent the axes of problem symmetry. It means that the axis x_1 coincides with the layer interface (see Fig. 1).

3. Governing equations and analytical solution

In fact, the motion of waves in each layer is described by the same equations used in the paper Adámek et al. (2015). Then $u_{1,n}(x_1, x_2, t)$ and $u_{2,n}(x_1, x_2, t)$ for $n = 1, 2$ are the functions of displacement components which we are looking for. Applying the Laplace transform in time domain, taking into account the zero initial conditions and introducing the complex functions $\bar{u}_{1,n} = \bar{u}_{1,n}(x_1, x_2, p)$ and $\bar{u}_{2,n} = \bar{u}_{2,n}(x_1, x_2, p)$ as the Laplace transforms of $u_{1,n}$ and $u_{2,n}$ ($p \in \mathrm{C}$), the transformed equations of motion can be written as

$$p^2\bar{u}_{1,n} = C_{11,n}^2\left(\frac{\partial^2\bar{u}_{1,n}}{\partial x_1^2} + \nu_{21,n}\frac{\partial^2\bar{u}_{2,n}}{\partial x_1\partial x_2}\right) + C_{12,n}^2\left(\frac{\partial^2\bar{u}_{1,n}}{\partial x_2^2} + \frac{\partial^2\bar{u}_{2,n}}{\partial x_1\partial x_2}\right), \tag{1}$$

$$p^2\bar{u}_{2,n} = C_{22,n}^2\left(\frac{\partial^2\bar{u}_{2,n}}{\partial x_2^2} + \nu_{12,n}\frac{\partial^2\bar{u}_{1,n}}{\partial x_1\partial x_2}\right) + C_{12,n}^2\left(\frac{\partial^2\bar{u}_{2,n}}{\partial x_1^2} + \frac{\partial^2\bar{u}_{1,n}}{\partial x_1\partial x_2}\right), \tag{2}$$

where the constants $\nu_{12,n}$ and $\nu_{21,n}$ denote the Poisson ratios of orthotropic material corresponding to the nth layer. For simplicity, the appropriate Poisson ratios of viscous and elastic elements in the material model are assumed to be equal. The complex functions $C_{11,n}(p)$, $C_{12,n}(p)$ and $C_{22,n}(p)$ present in (1)-(2) reflect the elastic and viscous properties of the material of each layer and are expressed by standard material parameters analogously as in Adámek et al. (2015).

Due to the symmetry of the problem with respect to the axis x_2, it is clear that the solution of the coupled system (1)-(2) can be found in the form of the following Fourier integrals:

$$\bar{u}_{1,n} = \frac{1}{\pi}\int_0^\infty A(\omega, x_2, p)\sin(\omega x_1)\,d\omega \quad \text{and} \quad \bar{u}_{2,n} = \frac{1}{\pi}\int_0^\infty B(\omega, x_2, p)\cos(\omega x_1)\,d\omega. \tag{3}$$

Introducing the expected solutions (3) into the system (1)-(2) and after some algebra, one obtains a system of two PDEs for the unknown Fourier spectra $A(\omega, x_2, p)$ and $B(\omega, x_2, p)$. The solution of such a system can be expressed in a general form

$$A(\omega, x_2, p) = P_n\,\mathrm{sh}(\Lambda_{1,n}x_2) + Q_n\,\mathrm{ch}(\Lambda_{1,n}x_2) + R_n\,\mathrm{sh}(\Lambda_{2,n}x_2) + S_n\,\mathrm{ch}(\Lambda_{2,n}x_2), \tag{4}$$

$$B(\omega, x_2, p) = L_{1,n}\left(P_n\,\mathrm{ch}(\Lambda_{1,n}x_2) + Q_n\,\mathrm{sh}(\Lambda_{1,n}x_2)\right) + L_{2,n}\left(R_n\,\mathrm{ch}(\Lambda_{2,n}x_2) + S_n\,\mathrm{sh}(\Lambda_{2,n}x_2)\right), \tag{5}$$

in which sh and ch stand for the hyperbolic functions sinh and cosh, respectively. The symbols $\Lambda_{1,n}$ and $\Lambda_{2,n}$ represent the roots of the characteristic biquadratic equation associated with the mentioned system of PDEs and they are dependent on the frequency ω and on p. The other quantities $L_{1,n}$ and $L_{2,n}$ depend on $\Lambda_{1,n}$ and $\Lambda_{2,n}$ and their definition can be deduced from the relations presented in Adámek et al. (2015).

At this moment, the Fourier spectra (4)-(5) are expressed in terms of eight unknown functions $P_n(\omega, p)$, $Q_n(\omega, p)$, $R_n(\omega, p)$ and $S_n(\omega, p)$ for n = 1, 2. These functions can be determined by using the boundary conditions of the problem. Based on the problem formulation made above, the boundary conditions can be formulated as follows:

$$\sigma_{22,2}(x_1,d,t)=\sigma(x_1,t), \quad \sigma_{12,2}(x_1,d,t)=0, \quad \sigma_{22,1}(x_1,0,t)=\sigma_{22,2}(x_1,0,t),$$
$$\sigma_{12,1}(x_1,0,t)=\sigma_{12,2}(x_1,0,t), \quad u_{1,1}(x_1,0,t)=u_{1,2}(x_1,0,t), \quad u_{2,1}(x_1,0,t)=u_{2,2}(x_1,0,t), \qquad (6)$$
$$u_{1,1}(x_1,-d,t)=0, \quad u_{2,1}(x_1,-d,t)=0.$$

Using the conditions (6) and the constitutive relations for appropriate stress components, a system of eight equations is obtained for P_n, Q_n, R_n and S_n. The analytical or numerical solution of this system can be found. Due to some overflow problems during the numerical computations performed in double precision in Matlab environment, the exact solution in a closed form was derived using the symbolic system Maple in this work. Substituting this solution into (4)-(5) and subsequently into relations (3), the resulting formulas for the Laplace transforms of displacement components are obtained. On the basis of these results, the Laplace transforms of other mechanical quantities, such as velocity or stress components, can be derived.

In the last step of the solving procedure, the inverse Laplace transform back to time domain needs to be performed. Based on the complexity of the resulting formulas and on the experiences gained by authors in their previous works (e.g. Adámek & Valeš, 2012), numerical approach to the inversion was chosen. In particular, an algorithm based on the combination of FFT and Wynn's epsilon accelerator was used. For details about this method see e.g. Cohen (2007).

4. Results and discussion

The evaluation of derived solution was made for a strip the material properties of which were estimated based on the parameters for composite lamina found in Soden et al. (1998). Concretely, following material parameters have been used: the material density ρ = 2250 kg m^{-3}; the Young moduli, shear modulus and the Poisson ratio of the alone-standing elastic element in the material model, see Sobotka (1984), $E_{0,1} = 35 \cdot 10^9$ Pa, $E_{0,2} = 11.584 \cdot 10^9$ Pa, $G_{0,12} = 4 \cdot 10^9$ Pa, $\nu_{0,12} = 0.278$; the Young modulus and the shear modulus of the second elastic element in the material model $E_1 = 18.48 \cdot 10^9$ Pa, $G_{12} = 1.83 \cdot 10^9$ Pa and the coefficients of normal and shear viscosities $\lambda_1 = \eta_{12} = 5 \cdot 10^4$ Pa s^{-1}. These parameters have been used in such a way to model a material with fibers oriented horizontally in the bottom layer of the strip (n = 1) and vertically in the upper layer (n = 2). The height of each layer was chosen as d = 20 mm. Finally, the function representing the external load applied to the upper strip edge (see Fig. 1) was assumed to be non-zero only for $x_1 \in \langle -h, h \rangle$ and had the form $\sigma(x_1,t) = -\sigma_a \cos(\pi/2 \cdot x_1/h)\, \mathrm{H}(t)$, where σ_a = 1 MPa, h = 2 mm and H(t) denotes the Heaviside function.

Fig. 2 shows a sample of results reached by the evaluation of the formulas derived (thick lines). The time histories of horizontal and vertical velocity components at two selected points (x_1 = 6, 10 mm) lying at the interface of the layers (x_2 = 0) are presented for $t \in \langle 0, 50 \rangle$ μs. To validate the procedure of analytical solution derivation and its evaluation, these results were compared to those obtained by FE simulation (thin dashed lines in Fig. 2). The simulation was performed in the professional FE code MSC.Marc/Mentat using linear isoparametric 4-node elements of basic size 0.4 × 0.4 mm. The Newmark algorithm with time step $2 \cdot 10^{-8}$ s was used for the integration in time domain. It is obvious from this comparison that a good agreement between both types of results was achieved and that the finite element model needs to be improved to reduce the oscillations of the time courses of velocity components for short times. This can be expected since relatively coarse mesh has been used to capture the waves of high frequencies. Additionally, it is clear from Fig. 2 that the major part of transient phenomena subsides in short times which is related to the material and geometric properties of the strip.

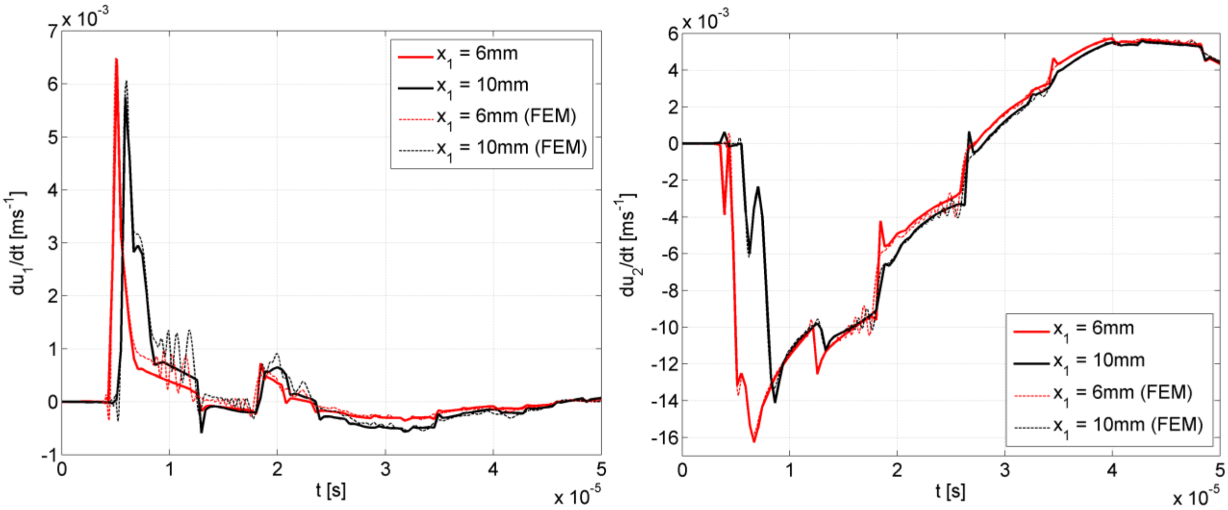

Fig. 2: Comparison of analytical (thick lines) and numerical (thin dashed lines) results for $x_2 = 0$ mm and for different values of x_1: (a) the horizontal velocity du_1/dt, (b) the vertical velocity du_2/dt.

5. Conclusions

This work presents the analytical solution for transient wave problem of an infinite viscoelastic strip composed of two specially orthotropic layers. Results for specific transversal load and free-fixed boundaries are obtained. Given the relatively general description of the strip material properties, this solution can be used for studying wave phenomena in strongly heterogeneous two-dimensional strips made of elastic, viscoelastic, isotropic or orthotropic layers. The advantage of this solution consists in the fact that it can be used not only for studying plane wave propagation through a layered structure as in most of existing works but also waves generated by a local load of impact character can be investigated. This can be utilised by the process of finding the optimal design of layered materials used for impact absorbers.

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