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# NUMERICAL SOLUTION OF A SECULAR EQUATION FOR RAYLEIGH WAVES IN A THIN SEMI-INFINITE MEDIUM MADE OF A COMPOSITE MATERIAL

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**Abstract:** *The traditional way of deriving the secular equation for Rayleigh waves propagating along the stress-free edge of a thin semi-infinite composite is presented. It means that it is necessary to find a general steady-state solution that vanishes at infinity. The secular equation is then obtained by vanishing of the surface traction at the stress-free edge. For the solution of such secular equation it is necessary to precompute some roots of characteristic quartic equation. The method shown in this paper, based on displacement formulation, leads to the so-called implicit secular equation. The numerical approach to the solution is shown.*

**Keywords: Rayleigh Waves, Composite Material, Secular Equation.**

## 1. Introduction

A thin semi-infinite composite structure is considered. The kind of the composite material in mind is one in which a matrix material is reinforced by strong stiff fibres which are systematically arranged in the matrix. The fibres are considered to be long compared to their diameters and the fibre spacing, and to be densely distributed, so the fibres form a substantial proportion of the composite. A composite of this kind for sufficiently long wavelength can be regarded as a homogeneous orthotropic material. It is also assumed that composite thickness is small compared to the shortest wavelength taken into account. Under these conditions one can consider the composite structure as an orthotropic solid in the state of plane stress. The traditional way of deriving the secular equation for Rayleigh-edge waves propagating in the direction of the $x_1$- axis in a thin semi-infinite composite $x_2 \geq 0$ is to find a general steady-state solution for the displacement components that vanishes at $x_2 = +\infty$. The secular equation is then obtained by vanishing of the surface traction at $x_2 = 0$. For the solution of such secular equation it is necessary to precompute some roots of characteristic quartic equation. The method shown in this paper (based on displacement formulation) leads to the so-called implicit secular equation. The details can be seen in the paper Cerv & Plesek (2013).

## 2. Preliminaries

We suppose that material and body axes of the 2D orthotropic linear elastic medium in the state of plane stress are denoted by $X_1$, $X_2$ and $x_1$, $x_2$ respectively. The third axis $x_3$ is identical with the material axis $X_3$ and constitutes axis of possible rotation (through an angle $\vartheta$) of the principal material axes $X_1$, $X_2$ from thre body axes $x_1$, $x_2$, see Fig. 1. Due to the plane stress it holds $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$. For considered material, the relationship between the stress $\sigma_{ij}$ and strain $\varepsilon_{ij}$ components is given by the formula (1), where $C_{ij} = C_{ij}(\vartheta)$ denote the elastic stiffnesses.

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$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} \tag{1}$$

The strain components $\varepsilon_{ij}$ are related to the displacement components $u_1$, $u_2$ through

$$2\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right). \tag{2}$$

The equations of motion, written in the absence of body forces, are

$$\sigma_{ij,j} = \rho \cdot \ddot{u}_i \,, \tag{3}$$

where $\rho$ is the mass density and the comma denotes differentiation with respect to $x_j$.

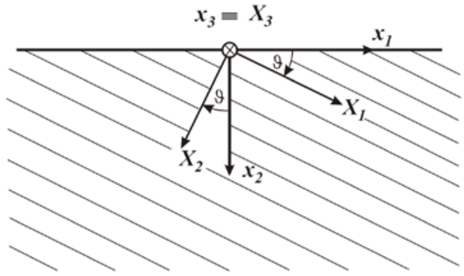


*Fig. 1: A thin semi-infinite orthotropic medium.*

## 3. Solution

Equations of motions in terms of displacements are considered, see Cerv & Plesek (2013). The solution to these equations is supposed in the form

$$\begin{aligned} u_1(x_1, x_2, t) &= U_{01} e^{q x_2} e^{\mathrm{i}k(x_1 - ct)} \,, \\ u_2(x_1, x_2, t) &= U_{02} e^{q x_2} e^{\mathrm{i}k(x_1 - ct)} \,, \end{aligned} \tag{4}$$

where $k$ is the wavenumber, $c$ the unknown velocity and $q$ a complex parameter dependent on $C_{ij}$, $c$ and $\rho$. It is stipulated that $\mathrm{Re}(q) < 0$. This solution represents a harmonic wave propagating in the positive direction of the $x_1$-axis. Boundary conditions can be stated as

$$\lim_{x_2 \to +\infty} u_1(x_1, x_2, t) = \lim_{x_2 \to +\infty} u_2(x_1, x_2, t) = 0 \,, \tag{5}$$

$$\sigma_{12} = \sigma_{22} = 0 \quad \text{at} \quad x_2 = 0 \,. \tag{6}$$

Substituting (4) into equations of motions one obtains a homogeneous system for displacement amplitudes, $U_{01}$, $U_{02}$ as

$$\begin{aligned} &[k^2 C_{11} - 2\mathrm{i}kqC_{16} - q^2 C_{66} - \rho k^2 c^2] U_{01} + [k^2 C_{16} - \mathrm{i}kq(C_{12} + C_{66}) - q^2 C_{26}] \, U_{02} = 0 \,, \\ &[k^2 C_{16} - \mathrm{i}kq(C_{12} + C_{66}) - q^2 C_{26}] U_{01} + [k^2 C_{66} - 2\mathrm{i}kqC_{26} - q^2 C_{22} - \rho k^2 c^2] U_{02} = 0. \end{aligned} \tag{7}$$

This system will have a nontrivial solution if and only if its determinant vanishes. This leads to a quartic characteristic equation in $p := \mathrm{i}q$, which may be written as

$$Ap^4 + Bk\,p^3 + Ck^2 p^2 + Dk^3 p + Ek^4 = 0 \,. \tag{8}$$

Due to the boundary condition (5) we are only interested in the roots satisfying $\mathrm{Im}(p) < 0$. The real coefficients $A, B, C, D, E$ are functions of velocity $c$ and material constants, see Cerv & Plesek (2013).

Using Ferrari's method it is possible to prove that it is sufficient to solve the quartic equation (8) for $k = 1$ only. All the roots must then be scaled by the true value of the wavenumber of interest, $k$. A quartic equation has four roots. In this case, the equation coefficients are real, hence the complex roots always come in conjugate pairs. It can be seen that the fulfilment of the boundary conditions at infinity $x_2 \to +\infty$ can only be guaranteed by the roots with $\mathrm{Im}(p) < 0$. Let us consider that $p_2,\ p_4,\ p_2 \neq p_4$ are such roots of (8) taken for $k = 1$. It turns out, without going into the details, that in the interval of velocities, where $\mathrm{Im}(p_2)$ and $\mathrm{Im}(p_4)$ are negative, the general solution to the equations of motion takes the form

$$\begin{aligned} u_1(x_1, x_2, t) &= [U_{01}^{(1)} e^{-\mathrm{i}kp_2x_2} + U_{01}^{(2)} e^{-\mathrm{i}kp_4x_2}]\, e^{\mathrm{i}k(x_1 - ct)} , \\ u_2(x_1, x_2, t) &= [U_{02}^{(1)} e^{-\mathrm{i}kp_2x_2} + U_{02}^{(2)} e^{-\mathrm{i}kp_4x_2}]\, e^{\mathrm{i}k(x_1 - ct)} . \end{aligned} \tag{9}$$

Now it is possible to express the stress components $\sigma_{12}$ and $\sigma_{22}$. It holds that

$$\begin{aligned} \sigma_{22} &= C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{26} 2\varepsilon_{12} , \\ \sigma_{12} &= C_{16}\varepsilon_{11} + C_{26}\varepsilon_{22} + C_{66} 2\varepsilon_{12} . \end{aligned} \tag{10}$$

The stress free boundary conditions at $x_2 = 0$, equation (6), yield another homogeneous system for displacement amplitudes $U_{01},\ U_{02}$ as

$$\begin{aligned} &[C_{12} + C_{22} p_2 D1 - C_{26}(p_2 + D1)]U_{01} + [C_{12} + C_{22} p_4 D2 - C_{26}(p_4 + D2)]U_{02} = 0 , \\ &[C_{16} + C_{26} p_2 D1 - C_{66}(p_2 + D1)]U_{01} + [C_{16} + C_{26} p_4 D2 - C_{66}(p_4 + D2)]U_{02} = 0 . \end{aligned} \tag{11}$$

This system will have a nontrivial solution if its determinant vanishes. This leads to a secular equation. The implicit secular equation may be written symbolically as (details may be seen in Cerv & Plesek (2013))

$$F(C_{ij}, \vartheta, \rho, c, p_j(C_{ij}, \vartheta, \rho, c)) = 0 . \tag{12}$$

## 4. Results

The determination of Rayleigh wave velocity by means of implicit secular equation (12) may be illustrated by the following example. Let us consider the thin composite SE84LV (Cerv et al., 2010). Let us also assume that the orientation of the principal material axes is given by $\vartheta = 45^\circ$.

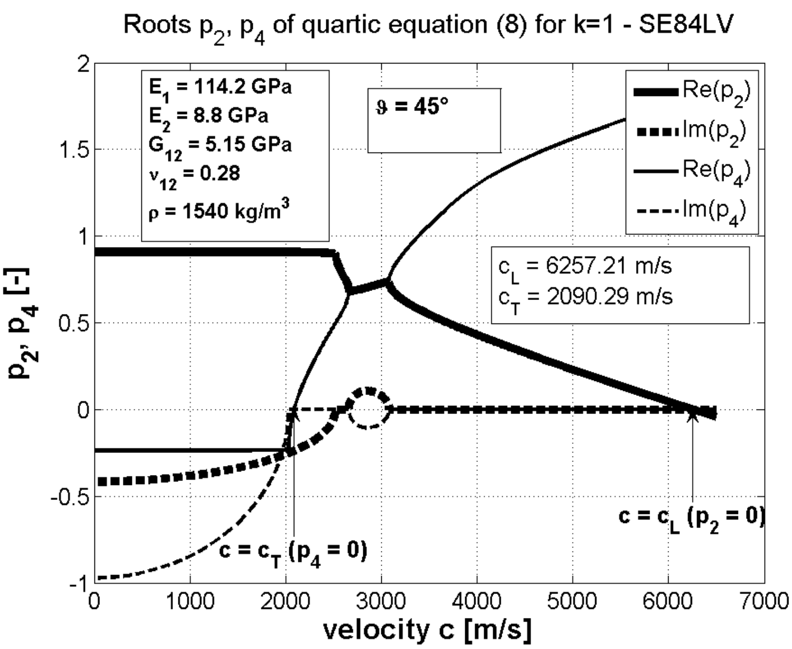


*Fig. 2: Roots $p_2, p_4$ of eq. (8) for $k = 1$ versus $c$.*

Before evaluating the left-hand side of the equation (12) as a function of velocity $c$ one has to compute the four roots $p_j$ of the quartic equation (8) taken for $k=1$. It can be seen that the fulfilment of boundary conditions at infinity $x_2 \to +\infty$ is guaranteed only by the roots $p_2, p_4$ with negative imaginary parts, see Fig. 2. A graph of the function $F$ is shown in Fig. 3. In an interval of speeds, where simultaneously $\mathrm{Im}(p_j) < 0$, the equation (12) has just one root $c_R = 1967.224$ m/s.

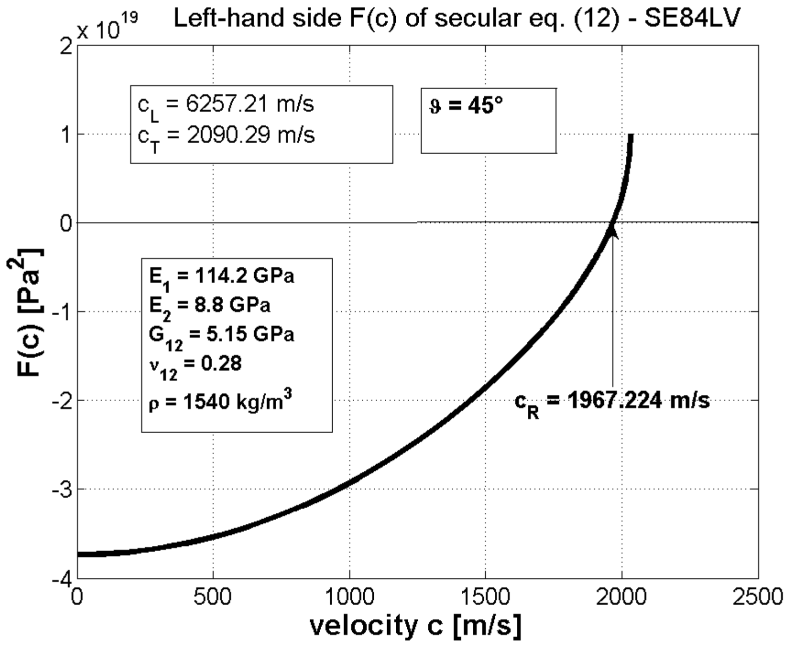


*Fig. 3: Left-hand side of eq. (12) versus velocity c.*

## 5. Conclusions

It has been confirmed, in accord with Ting's (2004), that the Rayleigh wave propagation exhibits no geometric dispersion. This means that the Rayleigh wave velocity is independent of frequency. In the case of orthotropic materials (thin composites) it has been found that Rayleigh wave velocity depends significantly, as with bulk waves, on the directions of the principal material axes. All numerical computations were performed in Matlab R2010b.


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