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APPROXIMATION OF THE MOVEMENT OF THE SPHERICAL PENDULUM

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Abstract: *The horizontally driven two-degrees of freedom spherical pendulum is an auto-parametric system which exhibits a wide variety of response types. Depending on the amplitude and frequency of the excitation the response can vary from stationary to chaotic regime. It has been already shown that the response can be described using time variable parameters of an inscribed ellipse. Such a description has a potential to close the gap between description of stationary, quasi-periodic and chaotic types of response. This study presents methodology and results of an analysis of the experimentally measured data providing the time dependent parameters of a twisting ellipse. The obtained results are discussed and some open problems are indicated.*

Keywords: Spherical pendulum, quasi-periodic response, data analysis.

1. Introduction

The spherical pendulum, which is kinematically driven in its suspension point, is a very popular auto-parametric system. In fact it is the simplest system, which exhibits complex response types, ranging from stationary periodic response to chaotic regime for certain driving conditions. Thus, many authors dealt various aspects of behaviour of the spherical pendulum until now, however, most of them are referencing papers by J. Miles (1962, 1984), where the weakly non-linear resonant response of a damped spherical pendulum is discussed. Among other papers dealing with aspects close to currently discussed topic can be mentioned, e.g., Petrov (2005) and Leung (2006), who analysed various types of kinematic or force excitation in suspension, where the harmonic excitation in the suspension point in both vertical and horizontal directions are considered. The experimental verification of the theoretical model was carried out by Tritton (1986).

Contribution of the authors to theoretical description of behaviour of the spherical pendulum comprises several publications; see, e.g., Náprstek & Fischer (2009, 2013). In first paper the non-linear mathematical model was introduced and its stability was analysed using the harmonic balance approach. In the latter paper, the concept of “virtual ellipse” was introduced and used for analysis of the quasi-periodic part of the response. The research was later supplemented by an experimental work by Pospíšil et. al (2014). The experimental setup was aimed to study the influence of uneven damping in both directions (lateral and transversal) with respect to direction of excitation. The current work returns to the concept of “virtual ellipse” and uses the experimental data to illustrate validity of the mathematical model.

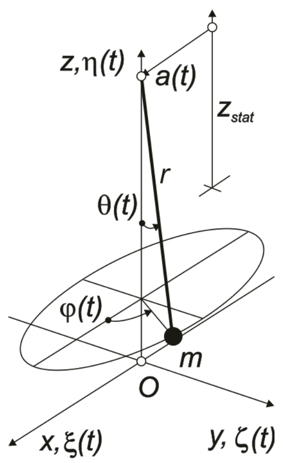

Fig. 1: Sketch of the system.

2. Theoretical model

The mathematical model of the spherical pendulum follows from the mechanical energy balance. Using the Hamilton principle and quadratic form of the Rayleigh function, a system of two Lagrange equations in Cartesian coordinates can be set up, for details see Náprstek & Fischer

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(2009) and Fig. 1. Neglecting terms of order $O(\varepsilon^6)$; $\varepsilon^2 = (\xi^2+\zeta^2)/r^2$, the approximate system reads:

$$\begin{aligned} &\ddot{\xi} + \frac{1}{2r^2}\xi\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\xi^2+\zeta^2) + 2\beta_\xi\dot{\xi} + \omega_0^2\xi\left(1+\frac{1}{2r^2}(\xi^2+\zeta^2)\right) = -\ddot{a} \\ &\ddot{\zeta} + \frac{1}{2r^2}\zeta\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\xi^2+\zeta^2) + 2\beta_\zeta\dot{\zeta} + \omega_0^2\zeta\left(1+\frac{1}{2r^2}(\xi^2+\zeta^2)\right) = 0 \end{aligned} \tag{1}$$

Here ξ stands for horizontal component of the response in direction of the excitation $a = a(t)$, while ζ describes the transverse motion. Symbols m, r represent mass and suspension length of the pendulum and β_ξ, β_ζ are the coefficients of linear viscous damping in the individual directions. The natural frequency of the corresponding linear pendulum is $\omega_0^2 = g/r$. The above equations are mutually independent if only the linear terms are considered; their interaction is given by the non-linear terms only.

With respect to assumption of the harmonic excitation and continuous character of the mathematical model and taking into account previously obtained analytical, numerical, and experimental results the system response can be approximated by following expressions:

$$\xi(t) = a_c(t)\cos\omega t + a_s(t)\sin\omega t\,, \quad \zeta(t) = b_c(t)\cos\omega t + b_s(t)\sin\omega t \tag{2}$$

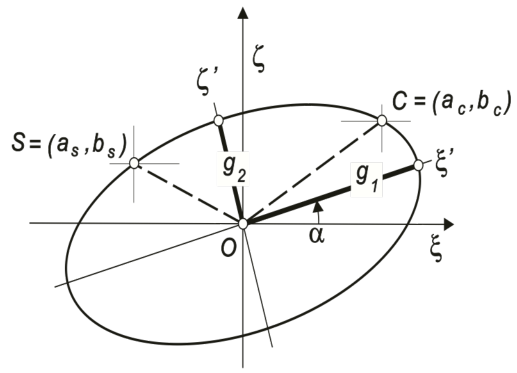

Fig. 2: Description of an ellipse in the xy plane

The partial amplitudes $a_c(t), a_s(t), b_c(t), b_s(t)$ in Eqs (2) are supposed to be the functions of a "slow" time and ω is the driving frequency of harmonic excitation. Enumeration of these amplitude functions is possible if the harmonic balance procedure is applied on the original system (1). As a result, a system of four first order ordinary differential equations can be obtained. Properties of the partial amplitudes reflect properties of different response types of the pendulum. The assumed solution (2) represents a parametric form of an ellipse in central position. It can be outlined in components ξ, ζ see Fig. 2, where $\alpha = \alpha(t)$ is orientation of principal axes; $g_1 = g_1(t)$ is the length of major axis and $g_2 = g_2(t)$- length of minor axis.

The three parameters α, g_1, g_2 determining properties of the inscribed ellipse are again functions of "slow" time. Variability of these functions is the same as variability of partial amplitudes (2). Their relation to the partial amplitudes is given as (see Náprstek & Fischer (2009) for details):

$$g_1 = \pm\frac{\sqrt{2S_A^4}}{\sqrt{R_A^2+\sqrt{R_A^4-4S_A^4}}}\,, \quad g_2 = \pm\frac{\sqrt{2S_A^4}}{\sqrt{R_A^2-\sqrt{R_A^4-4S_A^4}}}\,, \tag{3}$$

where R_A^2, S_A^2 are given as

$$R_A^2 = a_c^2 + a_s^2 + b_c^2 + b_s^2\,, \qquad S_A^2 = a_s b_c - a_c b_s\,. \tag{4}$$

The orientation of principal axes is given as:

$$\alpha(t) = \frac{1}{2}\arctan\frac{2(a_c b_c + a_s b_s)}{a_c^2 + a_s^2 - b_c^2 - b_s^2} \tag{5}$$

The curvature of the ξ, ζ trajectory considered as a planar curve is given as (Ruttler 2000):

$$k^2(t) = \frac{\left(\dot{\xi}\ddot{\zeta} + \ddot{\xi}\dot{\zeta}\right)^2}{\left(\dot{\xi}^2 + \dot{\zeta}^2\right)^3} \tag{6}$$

The approximation algorithm will be based on curvature of the experimental trajectory.

3. Approximation of experimental data

The ellipse in central position, which is supposed to approximate the experimental data (ξ_e, ζ_e), is fully described by three time dependent parameters α, g_1, g_2. The general expression reads:

$$\xi_e = g_1 \cos\alpha \cos\omega t - g_2 \sin\alpha \sin\omega t\,, \quad \zeta_e = g_1 \sin\alpha \cos\omega t + g_2 \cos\alpha \sin\omega t \tag{7}$$

The experimental (measured) data describe the trajectory of the pendulums bob in time; the driving frequency is ω known. In following it is assumed, that the measured data are filtered in such a way that they do not contain any spurious oscillations and thus that they represent a sufficiently smooth functions. This assumption eliminates cases when the trajectory degenerates into a line segment.

The basic idea of the algorithm is simple. In the first step, the maxima of the curvature of the experimental data have to be found. The radius vectors then allow determining orientation α of the ellipse and lengths of both major axis g_1 and minor axis g_2 at the corresponding time instants:

$$g_1 = \|(\xi_e, \zeta_e)\|\,, \quad g_2 = \sqrt{\frac{g_1}{k}}\,, \quad \alpha = \tan^{-1}\frac{\zeta_e}{\xi_e} \tag{8}$$

The relations (8) are valid for static ellipse; however, they provide sufficiently accurate initial values for subsequent optimization. This behaviour is illustrated in Figs. 3. Plot (A) shows the expected interpolation of the artificial data, which originate from discretization of a rotating ellipse (7), whereas the plot (B) uses real measured data. In both plots the dark solid lines represent the data to be approximated and dark dashed lines correspond to the ellipses inscribed in points of maximal curvature (shown as black dots). In case (A) the normal direction (dotted line) in the point of maximal curvature directs correctly to the origin. Thus, the approximation by an ellipse in central position is correct. For real measured data (B) is the incidence of the trajectory and approximating ellipse not as good. The normal direction in the point of maximal curvature (thin dotted lines) deviates from the radius vectors (thin dashed lines) significantly and the data trajectory is not symmetrical with respect to the normal. On the other hand, it seems that the ellipse in central position still provides acceptable approximation. Indeed, when the osculating circle (dotted circle) is shifted towards radius vector (dashed circle) in plot (B), it provides better approximation of the experimental trajectory than the osculating circle in the sense of least squares.

It seems that a similar approach can be used for points of minimal curvature; these points should directly provide length of the minor axis g_2. Unfortunately, because the trajectories are partly almost straight the search for a minimum leads to an ill-conditioned problem. On the other hand, the point of minimal curvature could be determined approximately dividing the angle between radius vectors of two adjacent points of maximal curvature or dividing the length of the arch. Because the curvature in such parts of the trajectory is low, the introduced error would be small and an additional point can improve the overall accuracy.

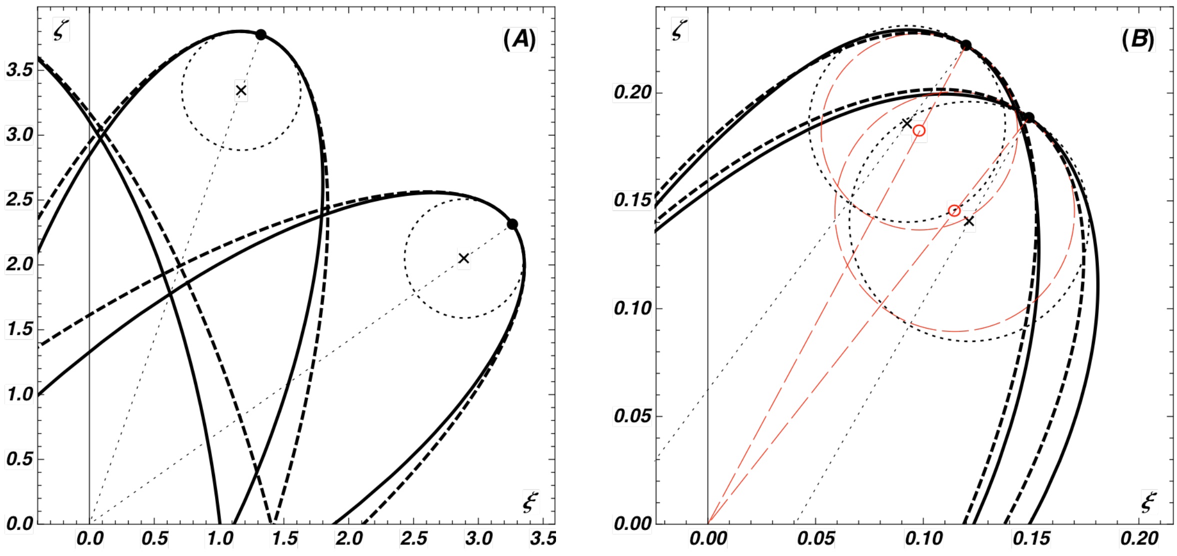

Fig. 3: Approximation of the rotating ellipse. (A) idealized case, (B) measured data. Thick solid curve - approximated data, thick dashed curve – inscribed ellipse, dotted circle - osculating circle

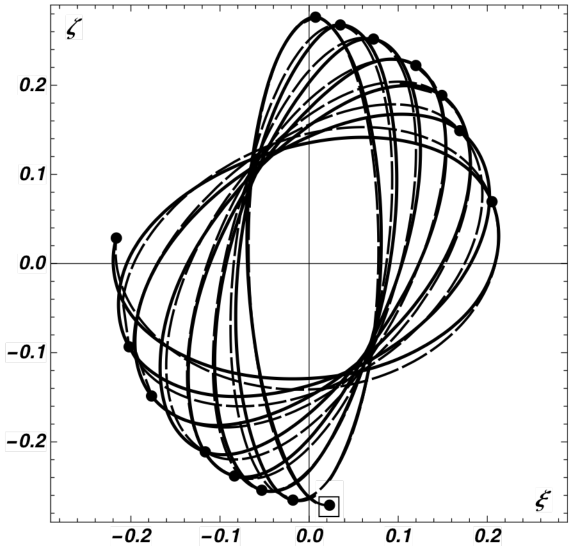

Fig. 4: Approximation of experimental data (solid) by the rotating ellipse (7), (dashed).

Figure 4 presents an example result of real approximation. The original data trajectory is shown as a solid curve, approximating ellipse (7) is dashed. The points of maximal curvature are indicated using black dots, starting point is denoted by a small square. It should be noted that the approximation in Figure 4 uses only curvatures computed in vertices of the ellipses. The numerically obtained values of α, g_1, g_2 were linearly interpolated as well as the computed "frequencies" $\omega = \pi/\Delta t$, which varied slightly around the driving frequency of the pendulum (Δt is the time difference between two instants of maximal curvature). The time coordinate had to be adjusted by introduction of a time (phase) shift in intervals between two instants of maximal curvature. No additional numerical optimization has been performed for results shown in Figure 4. For real usage, it would be worth to smoothen the obtained functions and improve the agreement using the least squares optimization method.

4. Conclusions

One of promising approaches for description of movement of the spherical pendulum is based on rotating ellipses. Presented contribution sketches out a procedure, which matches the experimental data to the theoretical model. The data are supposed to be sufficiently smooth or approximated by a smooth curve (e.g. spline). The approach based on observation of the curvature of the measured trajectory and usage of the extremal values to enumerate desired parameters proves to be usable. Using of non-extremal curvature values is feasible, but it brings some additional difficulties. Although it was not shown in this contribution, the similar approach is usable – after some modification – even for cases where the trajectory forms more complex geometrical curves (e.g., lemniscate). However, the degenerated cases (zero or infinite curvature) have to be dealt separately.

The presented approach is based purely on geometric curvature of the observed trajectory. It could be improved utilizing other geometrical relations. Compared to more traditional least squares interpolation approach the presented procedure can very quickly provide the approximate results. If necessary, these data could serve as a good initial approximation for subsequent optimization.

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