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DISPERSION PROPERTIES OF FINITE ELEMENT METHOD: REVIEW

R. Kolman[*], M. Okrouhlík[**], J. Plešek[***], D. Gabriel[****]

Abstract: *Review of the dispersion properties of plane square bilinear finite element used in plane elastic wave propagation problems is presented. It is assumed the grid (spatial) dispersion analysis and, further, the temporal-spatial dispersion analysis for explicit direct time integration based on the central difference method. In this contribution, the dispersion surfaces, polar diagrams and error dispersion graphs for bilinear finite element are depicted for different Courant numbers in explicit time integration. Finally, recommendation for setting the mesh size and the time step size for the explicit time integration of discretized equations of motion by the bilinear finite element method is provided.*

Keywords: Finite element method, dispersion properties, wave speed, explicit time integration.

1. Introduction

For accuracy analysis of the finite element method (FEM) (Hughes, 2000; Belytschko, 1983) in solving of wave propagation problems in solids is necessary to know dispersion properties of temporal-spatial semi-discretization. Generally, finite element (FE) solution is polluted by dispersion errors as an effect of spatial FE discretization [Belytschko, 1978; Mullen, 1982; Abboud, 1992] and by period elongation errors and numerical damping of direct time integration (Hughes, 2000; Belytschko, 1983). The dispersion errors are caused by differences of numerical wave speeds from the wave speeds in the ‘ideal’ continuum. Moreover, the FEM dispersion error is dependent on the frequency of propagating wave and on its orientation in a FE grid. For more information about dispersion properties of FEM see [Okrouhlík, 1993; Brepta, 1996; Červ, 1996; Plešek, 2010; Gabriel, 2010; Kolman, 2013; Kolman, 2015].

The temporal-spatial dispersion analysis of FEM in implicit and explicit direct time integration has been studied in [Schreyer, 1983; Marfurt, 1984]. The central difference method with the diagonal (lumped) mass matrix [Dokainish, 1989] is widely utilized in explicit time integration, while the Newmark method with the consistent mass matrix [Newmark, 1959] is employed in implicit computations. In principle, implicit computation needs a much larger computational effort per time step due to the solving a linear equation system. On the other hand, explicit methods with a diagonal mass matrix require a vector solver only, but mostly they are merely conditionally stable. Thus, the time step size must satisfy the stability limit, [Park, 1977]. In wave propagation and impact problems, the explicit time integration is preferred, therefore we focus only on the central difference method [Dokainish, 1989].

2. Wave propagation in an elastic unbounded domain

It is known that two types of elastic waves propagate through an elastic unbounded domain (Achenbach, 1973). The first wave is the longitudinal one propagating with the wave speed given by $c_1 = \sqrt{(\Lambda + 2G)/\rho}$ and the second one is the transverse wave with the wave speed $c_2 = \sqrt{G/\rho}$, where Λ and G are the Lamé’s constants and ρ is the mass density.

[*] Ing. Radek Kolman, PhD.: Institute of Thermomechanics, Czezh Academy of Science, Dolejškova 1402/5; 182 00, Prague; CZ, kolman@it.cas.cz

[**] Prof. Ing. Miloslav Okrouhlík, CSc.: Institute of Thermomechanics, Czech Academy of Science, Dolejškova 1402/5; 182 00, Prague; CZ, ok@it.cas.cz

[***] Ing. Jiří Plešek, CSc.: Institute of Thermomechanics, Czech Academy of Science, Dolejškova 1402/5; 182 00, Prague; CZ, plesek@it.cas.cz

****Ing. Dušan Gabriel, PhD.: Institute of Thermomechanics, Czech Academy of Science, Dolejškova 1402/5; 182 00, Prague; CZ, gabriel@it.cas.cz

In dispersion analysis, we assume a plane wave solution for components of a displacement field in the form $u_i = U_i \exp[\mathrm{i}\, k\,(\mathbf{x}\,\mathbf{p} \pm c\,t)]$, $i = 1,2$, where $\mathrm{i} = \sqrt{-1}$, $\mathbf{k} = (k_x, k_y)$ is the wave vector, $k = \sqrt{k_x^2 + k_y^2}$ is the wavenumber, $\mathbf{x}$ is the position vector, $\mathbf{p}$ is the unit vector describing direction of wave propagation, c is the phase speed, t is the time and U_i is the *i*-th component of displacement vector. Relationship for the angular velocity of wave ω is given by $\omega = k\,c$ and the wavelength λ is computed as $\lambda = 2\pi / k$. The positions of nodes with the indexes m, n in the bilinear FE mesh are prescribed as $x_m = mH$, $y_n = nH$, where H marks the edge length of a bilinear finite element. The components of the unit vector $\mathbf{p}$ are defined by the angle θ: $p_x = \cos\theta$, $p_y = \cos(\pi/2 - \theta)$ (Fig. 1).

A dimensionless time step size is defined by the Courant number as $Co = \Delta t c_1 / H$, where Δt is the time step size.

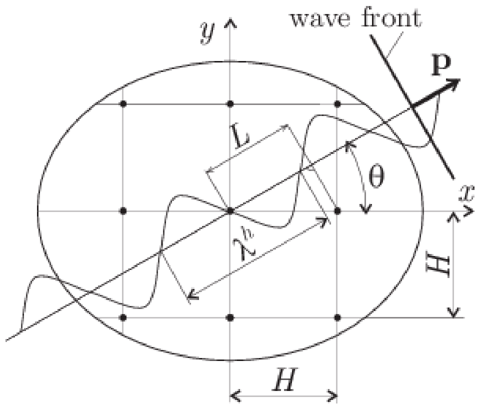

Fig. 1: A plane infinite bilinear regular finite element mesh and plane wave inclined by angle θ.

3. Dispersion properties of the bilinear finite element method

Results of spatial and temporal-spatial dispersion analysis of the bilinear finite element and for the explicit time integration based on the central difference method with the lumped mass matrix are presented on Figs. 2, 3 and 4. The results are depicted for several Courant numbers, where the value $Co = 1.0$ corresponds to the critical time step size for the bilinear finite element with the lumped mass matrix [Park, 1977].

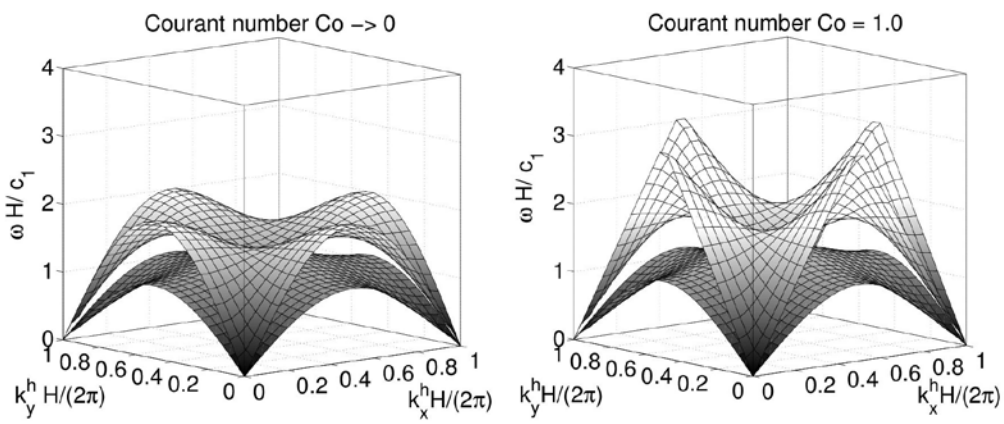

Fig. 2: Temporal-spatial dispersion relations of a plane square bilinear finite element with the lumped mass matrix for Courant numbers: Co → 0 and Co = 1.0.

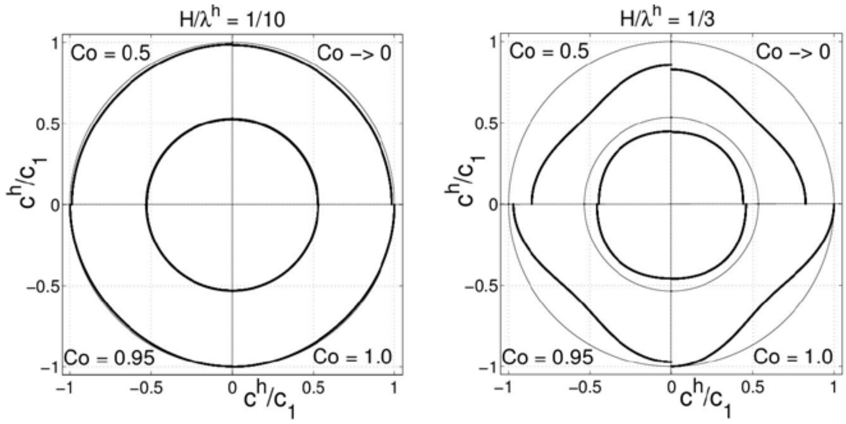

Fig. 3: Polar temporal-spatial dispersion diagrams of a plane square bilinear finite element with the lumped mass matrix for Courant numbers Co={→0, 0.5, 0.95, 1.0} for $H/\lambda^h = 1/10$ and $H/\lambda^h = 1/3$.

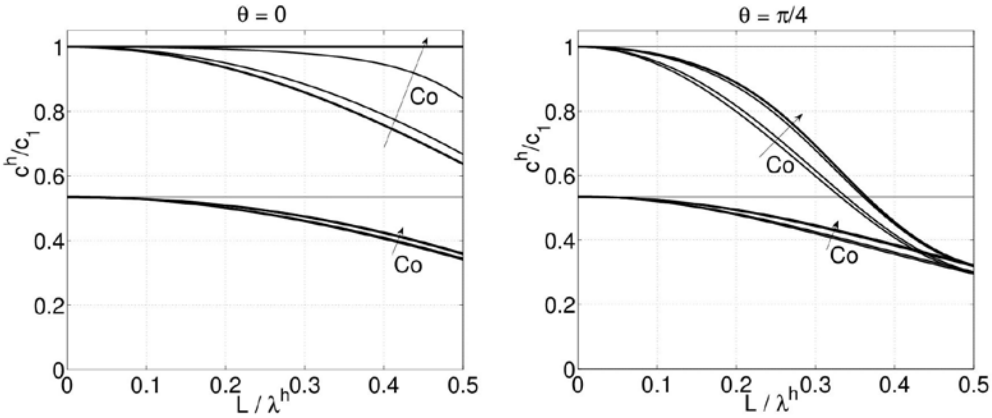

Fig. 4: Dispersion errors in phase velocity of a plane square bilinear finite element with the lumped mass matrix for Courant numbers Co = {→ 0, 0.5, 0.95, 1.0} for propagation directions given by angles $\theta = 0$ and $\theta = \pi/4$.

Based on the temporal-spatial dispersion analysis of the bilinear finite element, we can see an effect of improving dispersion errors for longitudinal waves for the critical time step (the Courant number close to $Co = 1.0$) and for wave direction given by $\theta = 0$. On the other hands, this effect is not seen for the transverse waves. Therefore, the best choice for the time step size with respect to accuracy of the linear FEM and explicit time integration is the time step size given by the stability limit [Park 1977].

4. Conclusions

Based on the dispersion analysis of the bilinear finite element method, the edge length of the finite element H is recommended to choice so that it is satisfied the conditions $H_{\max} \le \lambda / 10$. The wavelength can be estimated as $\lambda = c_2 / f_{\max}$, where $f_{\max}$ is the maximal loading frequency in Hz.

Further, it was shown that dispersion errors in explicit FE modelling can be improved only for longitudinal waves, where we should integrate with the critical time step size by the central difference method. In principle, the dispersion errors for transverse waves are independent of a choice of the stable time step size. Other way how to eliminate dispersion errors also for transverse waves is to use the partitioned wave explicit scheme presented in [Kolman, Cho, Park, 2015].

This contribution is dedicated to the seventy-fifth birthday of Professor Miloslav Okrouhlík.

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