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CENTRAL DIFFERENCE METHOD APPLIED FOR THE BEAM RESTED ON NONLINEAR FOUNDATION (PROGRAMMING AND EVALUATION OF RESULTS)

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Abstract: *This work presents practical applications of experimental and numerical approaches in the solution of straight beams on elastic foundations. There are tangent-linear, nonlinear (i.e. linear + arcus tangent) and secant-linear approximations for dependencies of distributed reaction forces on deflection in the foundation. For solutions of nonlinear problems of mechanics, the Central Difference Method is applied in combination with the Newton Method. The results acquired by linear/nonlinear solutions are evaluated and compared.*

Keywords: Elastic foundation, Beam, Central difference, Nonlinearity, Programming.

1. Introduction

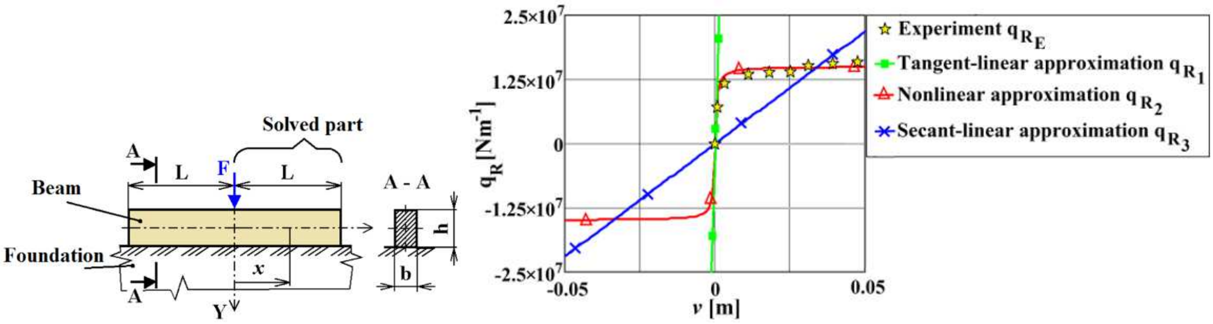

Fig. 1. Solved beam of length 2L resting on an elastic foundation and dependence of reaction force on deflection (i.e. foundation load-settlement behaviour).

Beams on elastic foundations are frequently used in engineering; see (Frydrýšek et al., 2013). This work is a continuation of our previous works. The symmetrical beam of length 2L = 2×12.045 m with cross-section b×h (b = 0.2 m, h = 0.4 m) is resting on an elastic foundation. The beam is loaded by force F = 7×10^6 N. The modulus of elasticity of the beam is $E = 2\times10^{11}$ Pa and the principal quadratic moment of the beam cross-section is $J_{ZT} = \frac{\mathrm{bh}^3}{12}$. The nonlinear behaviour $\mathrm{q_R} = \mathrm{q_R}(v)$ [Nm^{-1}] of the distributed reaction force on deflection v [m] in the foundation was approximated by tangent-linear $\mathrm{q_{R1}} = k_1 v = 1.7422\times10^{10} v$, non-linear $\mathrm{q_{R2}} = k_1 v + k_a \mathrm{arctg}(c_a v) = 5.21\times10^5 v + 9.52\times\ 10^6 \mathrm{arctg}(1.83\times10^3 v)$ and secant-linear $\mathrm{q_{R3}} = k_1 v = 4.3866\times10^8 v$ functions. The nonlinear approximation $\mathrm{q_{R2}}$ fits the best with experiment. The solution deals with nonlinear differential equation $\frac{d^4 v}{dx^4} - \frac{k_1 v + k_a \mathrm{arctg}(c_a v)}{E J_{ZT}} = 0$. In the cases of linear solutions (i.e. tangent-linear and secant-linear approximations), the function $k_a \mathrm{arctg}(c_a v) = 0$. The parameters k_1, k_a and c_a were acquired from measurements $\mathrm{q_{RE}}$ by curve fitting; see (Frydrysek et al., 2013).

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2. Central Difference Method (CDM) – Study case of Beam on Elastic Foundation

According to CDM (see Fig. 2 and (Frydrýšek et al., 2013), i.e. discretization of a nonlinear differential equation), the system of n+1 nonlinear equations for approximations q_{R1}, q_{R2} and q_{R3} can be derived.

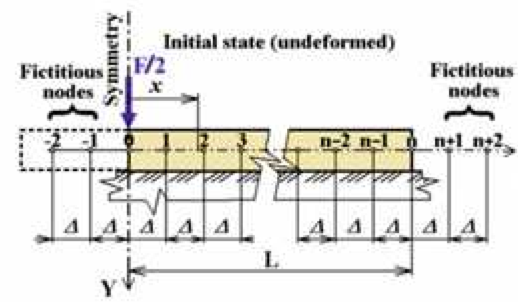

Fig. 2. CDM - Divisions of the beam (study case).

Hence, the system of nonlinear equations includes boundary conditions,

$$[\mathrm{M}]\{v\} + a_2 \mathrm{arctg}(c_a\{v\}) - \{\mathrm{b}\} = \{0\}, \tag{1}$$

where
$$[\mathrm{M}] = \begin{bmatrix} c & -8 & 2 & 0 & 0 & 0 & 0 & \dots & 0 \\ -4 & 7+a_1 & -4 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & -4 & c & -4 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -4 & c & -4 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 1 & -4 & c & -4 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & -4 & c & -4 & 1 \\ 0 & \dots & 0 & 0 & 0 & 1 & -4 & 5+a_1 & -2 \\ 0 & \dots & 0 & 0 & 0 & 0 & 2 & -4 & 2+a_1 \end{bmatrix}, \ \{b\} = \begin{Bmatrix} b \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \ \{v\} = \begin{Bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{Bmatrix},$$

$\Delta = \frac{\mathrm{L}}{\mathrm{n}}$ (see Fig. 2), $b = \frac{\mathrm{F}\Delta^3}{EJ_{ZT}}, a_1 = \frac{k_1\Delta^4}{EJ_{ZT}}, a_2 = \frac{k_a\Delta^4}{EJ_{ZT}}, c = 6 + a_1$.

3. Newton Method Iterative Approach

The nonlinear equations (1) can be solved iteratively via Newton Method as

$$\left\{{}^{\langle \mathrm{j}+1\rangle}v\right\} = \left\{{}^{\langle \mathrm{j}\rangle}v\right\} - \left[{}^{\langle \mathrm{j}\rangle}\mathrm{J}\right]^{-1}\left\{[\mathrm{M}]\left\{{}^{\langle \mathrm{j}\rangle}v\right\} + a_2\mathrm{arctg}\left(c_a\left\{{}^{\langle \mathrm{j}\rangle}v\right\}\right) - \{\mathrm{b}\}\right\}. \tag{2}$$

Where vectors of displacement $\left\{{}^{\langle \mathrm{j}\rangle}v\right\}$ and $\left\{{}^{\langle \mathrm{j}+1\rangle}v\right\}$ are old and new iterations and $\left[{}^{\langle \mathrm{j}\rangle}\mathrm{J}\right]$ is the Jacobian defined by

$$\left[{}^{\langle \mathrm{j}\rangle}\mathrm{J}\right] = \left(\frac{\partial\left[[\mathrm{M}]\left\{{}^{\langle \mathrm{j}\rangle}v\right\} + a_2\mathrm{arctg}\left(c_a\left\{{}^{\langle \mathrm{j}\rangle}v\right\}\right) - \{\mathrm{b}\}\right]}{\partial\,{}^{\langle \mathrm{j}\rangle}v_{\mathrm{k}}}\right)_{\mathrm{k}=0,1,2,\dots,\mathrm{n}}, \tag{3}$$

Matrix $\left[{}^{\langle \mathrm{j}\rangle}\mathrm{J}\right]$ is changing for each iteration ⟨j⟩. However, matrix $\left[{}^{\langle \mathrm{j}\rangle}\mathrm{J}\right]$ is "similar" to matrix [M] (i.e. both are sparse and their distinctions are only in main diagonals of these matrices) which is suitable for programming. For more information, see (Frydrýšek et al., 2014; Michenková et al., 2014).

4. Evaluation of Acquired Results

Some basic results for a long beam are presented in Fig. 3 (i.e. dependencies for deflection, slope, bending moment, shear force and reaction force on length coordinate *x* for q_{R1}, q_{R2} and q_{R3} approximation of foundation). The slope of the beam is defined as $\frac{dv}{dx}$, the bending moment is defined as $M_o = -EJ_{ZT}\frac{d^2v}{dx^2}$, the shear force is defined as $T = -EJ_{ZT}\frac{d^3v}{dx^3}$ and the reaction force in the foundation is defined via q_{R1}, q_{R2} and q_{R3} functions. The distinctions between each type of foundation are evident. For presented inputs, the tangent-linear approximation q_{R1} and nonlinear approximation q_{R2} of foundation give nearly the same results (i.e. good agreement with experiment noted in chapter 1). However the secant-linear approximation q_{R3} gives unreal results. This is caused by “quite small” loading force F = 7×10[6] N.

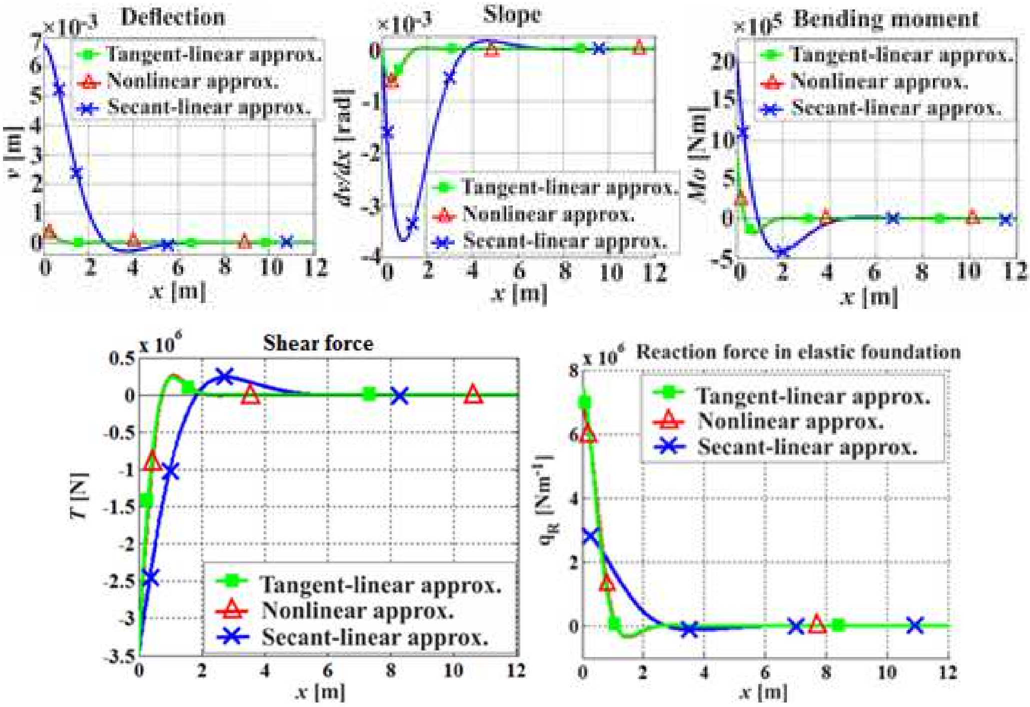

Fig. 3. Dependence of deflection, slope, bending moment, shear force and distributed reaction force on coordinate $x\epsilon\langle 0; L\rangle$ of the beam for different types of foundation approximations (results acquired by CDM with MATLAB).

However, increasing of external force F brings higher influence of nonlinearity (i.e. bigger differences between linear and nonlinear solutions). Hence, dependencies of maximum values of displacement v_{MAX} [m], bending moment $M_{o_{\mathrm{MAX}}}$ [Nm] and reaction forces q_{RMAX} [Nm^{-1}] on force F are presented in Fig. 4. These figures were printed for the same beam rested on elastic linear/nonlinear foundation.

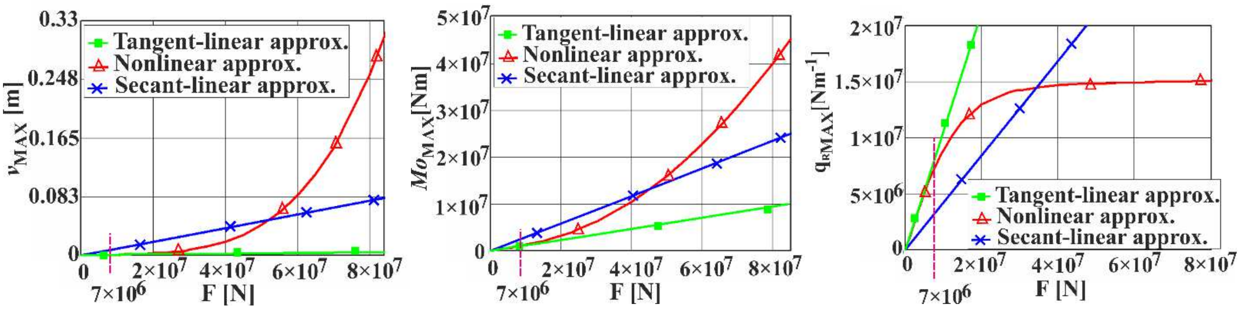

Fig. 4. Dependence of maximum deflection, maximum bending moment and maximum reaction force on external force for the beam on different types of foundation approximations (results acquired by CDM with MATLAB).

Now, the distinctions between each type of foundation approximations (i.e. influences of nonlinearities) are evident. As it was mentioned, the nonlinear approximation q_{R2} is the best approximation of the reality. Therefore, for the small deflections of foundation fit well tangent-linear approximations q_{R1} (i.e. for $F \in \langle 0; 3 \times 10^7\rangle$ N). Otherwise, for larger deflections fits well secant-linear approximation q_{R3} (i.e. for $F \in \langle 4 \times 10^7; 6 \times 10^7\rangle$ N). However, nonlinear approximation q_{R2} fits well for all cases of deflections (i.e. for $F \in \langle 0; 8 \times 10^7\rangle$ N). To put that into context, in Fig. 4, there is marked the value of loading force $F = 7{\times}10^6$ N which is connected with the solution presented in Fig. 3.

5. Future Application

The use of an elastic foundation including nonlinearities is a suitable way of performing numerical/experimental modelling of engineering problems. For example modelling of external fixators designed for the treatment of complicated bone fractures, modelling of femoral screws designed for the treatment of "collum femoris" fractures (see Fig. 5 and reference Frydrýšek et al., 2013).

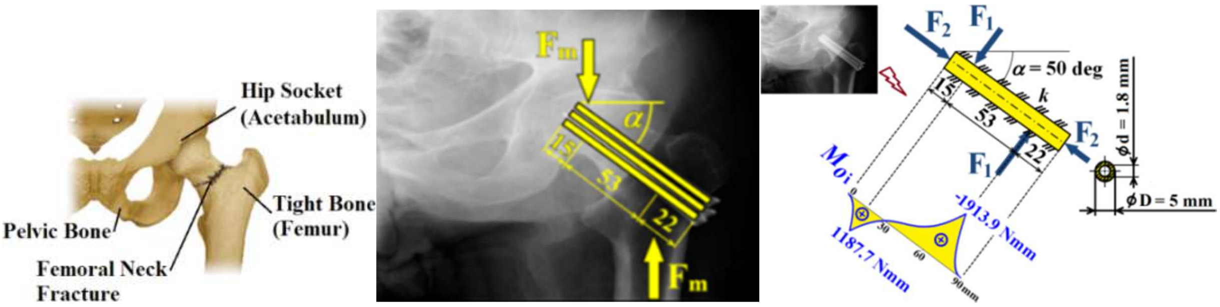

Fig. 5. Examples of beams on elastic foundations in the field of biomechanics (collum femoris fracture and its treatment and numerical solutions of femoral cannulated screw).

The derivation, rapid solutions and application of our own simple numerical model based on CDM open up new possibilities for further applications using a stochastic approach (i.e. millions of solutions with random inputs and outputs can be easily simulated). Therefore, the application of the CDM + probabilistic approach connected with the probabilistic reliability assessment of femoral screws is the main focus of the future work, see (Marek et al., 1995).

6. Conclusions

The measured material properties of the elastic foundation were evaluated and approximated in three ways (via easy bilateral tangent-linear $q_{R1} = k_1 v = 1.7422 \times 10^{10} v$, complicated but complex bilateral nonlinear $q_{R2} = k_1 v + k_a \mathrm{arctg}(c_a v) = 5.21 \times 10^5 v + 9.52 \times 10^6 \mathrm{arctg}(1.83 \times 10^3 v)$ and easy secant-linear $q_{R3} = k_1 v = 4.3866 \times 10^8 v$ functions). Beams on elastic linear and nonlinear foundations were solved via the CDM and iterative Newton Method using MATLAB software. The iterative approach is necessary for nonlinear solutions. From the results, it is evident that the nonlinear approximation for the behaviour of an elastic foundation fits very well with experiments and gives the best results. However, the application of the CDM and iterative Newton Method (i.e. solutions of nonlinear problems) is possible, though complicated (i.e. time-consuming). Tangent/secant-linear approximations of the elastic foundations behaviour give worse results, though acceptable in some cases. It could be dangerous to place blind faith in the easy linear approximation of the foundation.

In references (Frydrýšek et al., 2013; Frydrýšek et al., 2014; Michenková et al., 2014) are presented other approximations and solutions of similar beam rested on elastic foundation with different behaviour. Possible future improvements are explained too. The application of CDM is quite easy, comprehensible and suitable for beam structures. Numerical approaches used in this article could be applied in many engineering solutions.

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