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NON-LINEAR NORMAL MODES IN DYNAMICS-
-CONTINUOUS SYSTEMS

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Abstract: *The paper presents a continuation of an effort started last year, when the authors briefly informed about the Non-linear normal modes (NNM) concerning the version dealing with discrete systems. Although many features of the continuous formulation from the mathematical viewpoint are similar to the discrete case, a couple of specifics should be highlighted from the viewpoint of a real applicability of this tool to investigate particular dynamic systems. Three approaches are mentioned in the paper and the Galerkin-Petrov based procedure is outlined in more details. As a particular subject the cantilever prismatic beam is discussed. Non-linear normal modes for several amplitudes are shown to demonstrate the dependence of their shapes on the actual effective amplitude. Comparison with adequate linear counterpart is done.*

Keywords: Nonlinear dynamic systems, non-linear normal modes, discretization, multi-scale method

1. Introduction

The concept of the natural mode decomposition is very popular in linear area for many reasons, which are widely known. As it has been reported earlier this concept has been generalized to non-linear systems, for instance Nayfeh (1994), Shaw S.W. (1994) and many others. A large review can be found in pioneering papers by Vakakis and in the monograph by Vakakis et al. (1996). While most of the useful properties of linear modes cannot be reflected in the nonlinear area, invariant motions on a two-dimensional manifold can be found. Such motions have been named as nonlinear mode motions and the relevant method as Nonlinear Normal Mode (NNM) decomposition. This discipline is developing roughly some 30 years, although older papers dealing with this idea have appeared since early 60s, e.g., Rosenberg (1960, 1966), etc. Nevertheless, a lot of papers have been published during last years dealing with various special aspects of NNM, see for instance Kerschen (2009) - numerical implementations, Lenci (2007) - systems with internal resonance, etc.

In the previous paper the authors informed briefly about the version concerning discrete systems, see Náprstek (2015) with reference to the first paper appeared in Czech literature related with NNM, see Byrtus (2012). Regarding a continuous system, a couple of approaches are considered in the literature being based on methods commonly used in nonlinear dynamics:

(i) Time coordinate is anticipated in a form of a suitable periodic function $\varphi(t)$ (not necessarily harmonic) and the unknown displacement is sought in the form of $w(x,t) = v(x)\varphi(t)$. This expression is substituted into the governing system and then the method of the harmonic balance of an adequate level is used. This results in a boundary value problem for $v(x)$ which can serve to obtain a set of NNMs.

(ii) The Galerkin-Petrov procedure is applied to discretize the original problem. The displacement can be written in a form:

$$w(x,t) = \sum_{i=1}^{N} v_i(x)\varphi_i(t) \tag{1}$$

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where $v_i(x)$ are the linear undamped mode shapes. Then the expression (1) is substituted into the original nonlinear system and the Galerkin-Petrov procedure of orthogonalization in the space of admissible functions is subsequently performed. This results to a system of coupled ODEs for unknown time coordinates $\varphi_i(t)$. Thus, the NNMs are formulated as time variable linear combinations of linear modes. As the linear eigen modes are used as coordinate functions, all boundary conditions are implicitly fulfilled and optimal convergence is guaranteed.

(iii) NNMs and amplitude dependent eigen frequencies can be determined by applying the method of multiple scales directly to the governing partial differential equation and boundary conditions. Basically similar procedure as it is known in the perturbation method is performed.

As a particular case we treat a strait prismatic console with axial nonlinear effects due to the axial load and other effects depending on static and dynamic processes, see for instance Crespo da Silva & Glyn (1978). Following partial differential equation can be written:

$$\begin{aligned} &\ddot{w}(x,t)+w''''(x,t)+Q[w(x,t)]=0, \\ &w(0,t)=0,\ \ w'(0,t)=0,\ \ w''(1,t)=0,\ \ w'''(1,t)=0, \\ &Q[w(x,t)]=[w'(x,t)\,(w'(x,t)w''(x,t))']'+\left[w'(x,t)\int_1^x\int_0^\xi(\dot{w}(\zeta,t)'^2+w'(\zeta,t)\ddot{w}'(\zeta,t)\,d\zeta\,d\xi\right]' \end{aligned} \tag{2}$$

where $w(x,t)$ denotes the vertical displacement of the console and $Q[w(x,t)]$ represents the non-linear part of the equation comprising effects of the deformed system, the symbols $\bullet\,',\ \dot{\bullet}$ indicate differentiation with respect to space or time, respectively. Concerning $Q[w(x,t)]$, a lot of more complex or simpler variants are available in literature. Their form is dependent on nonlinear effects which are important from the viewpoint of respective bifurcations and post-critical processes.

2. Discretized system

Let us demonstrate application of the 2nd solution process, see (ii) and Eq. (1). We consider the series:

$$w(x,t)=\sum_{i=1}^{\infty}v_i(x)\,\varphi_i(t) \tag{3}$$

where $\varphi_j(t)$ are the relevant time coordinates and $v_i(x)$ are linear mode shapes of Eq. (2):

$$\begin{aligned} &v_i(x)=\cosh\lambda_i x+\cos\lambda_i x+\beta_i(\sin\lambda_i x+\sinh\lambda_i x) \\ &\beta_i=\frac{(\cosh\lambda_i+\cos\lambda_i)}{(\sin\lambda_i+\sinh\lambda_i)},\qquad 1+\cosh\lambda_i\cos\lambda_i=0 \end{aligned} \tag{4}$$

Substituting Eq. (3) into (2) and performing Galerkin's operations we obtain differential system for $\varphi_i(t)$:

$$\begin{aligned} &\ddot{\varphi}_i+\omega_i^2\varphi_i+R_i(\varphi_i,\dot{\varphi}_i,\ddot{\varphi}_i)=0 \\ &R_i=\int_0^\xi v_i(\xi)\cdot Q\left[\sum_{i=1}^n v_j(\xi)\varphi_j(t),\sum_{i=1}^n v_j(\xi)\dot{\varphi}_j(t),\sum_{i=1}^n v_j(\xi)\ddot{\varphi}_j(t)\right]d\xi \end{aligned} \tag{5}$$

The term $Q\,[\cdot,\cdot,\cdot]$ in (5) symbolically comprises value and two time derivatives of $w(x,t)$. We approximate the nonlinear parts R_i in Eq. (5) employing decomposition:

$$\begin{aligned} &R_i(\varphi_j,\dot{\varphi}_j)=(g_{1ij}-\omega_j^2 g_{2ij})\varphi_j^3+g_{2ij}\varphi_j\dot{\varphi}_j^2+\dots, \\ &\text{where:}\quad g_{1ij}=\int_0^1 v_i(\xi)\,Q_1(v_j(\xi))\,d\xi,\quad g_{2ij}=\int_0^1 v_i(\xi)\,Q_2(v_j(\xi))\,d\xi, \\ &Q_1(v_j(\xi))=(v_j'(v_j'v_j'')')',\qquad Q_2(v_j(\xi))=(v_j'\int_1^\xi\int_0^\zeta v_j'^2\,d\chi\,d\zeta)' \end{aligned} \tag{6}$$

It can be shown, cf. Nayfeh (1994), that movement of the system in the i-th eigen mode can be written as:

$$w_i(x,t) = v_i(x)\varphi_i(t) + \sum_{i \neq j} v_j(x)[\Gamma_{1ij}\varphi_j^3(t) + \Gamma_{2ij}\varphi_j(t)\dot{\varphi}_j^2(t) + \dots]$$

$$\Gamma_{1ij} = [(7\omega_j^2 - \omega_i^2)g_{1ij} - (5\omega_j^2 - \omega_i^2)\omega_j^2 g_{2ij}] / \Delta_{ij} \tag{7}$$

$$\Gamma_{2ij} = [6g_{1ij} - (3\omega_j^2 + \omega_i^2)g_{2ij}] / \Delta_{ij}, \qquad \Delta_{ij} = (\omega_j^2 - \omega_i^2)(9\omega_j^2 - \omega_i^2)$$

We avoid until now cases $\omega_i \approx \omega_j$, $\omega_i \approx 3\omega_j$ leading to internal resonance. They produce homoclinic orbits and should be treated separately at the center manifold. Such cases occur in practice and cannot be omitted. However, a special study should be devoted to these effects, in particular regarding continuous systems. Nevertheless, the structure of Eq. (7) exhibits that every nonlinear eigen mode captures the contribution of all other linear modes. Hence the modal motion time history is given by expression:

$$\varphi_i(t) = a_i \cos(\omega_{Ni}t + \beta_{i0}) + \frac{g_{1ii} - 2\omega_i^2 g_{2ii}}{32\omega_i^2} a_i^3 \cos(3\omega_{Ni}t + 3\beta_{i0}) + \mathrm{K} \;,$$

$$\omega_{Ni} = \omega_i + \frac{1}{8\omega_i}(3g_{1ii} - 2\omega_i^2 g_{2ii})a_i^2 + \mathrm{K} \tag{8}$$

where ω_{Ni} is the nonlinear eigen frequency of the *i*-th mode. Depending on the initial conditions are a_i, β_{i0} constants, which represent first approximation to the amplitude and phase of the motion.

3. Numerical experiments

In order to illustrate the above considerations we outlined first three nonlinear modes for different amplitudes and compared them with adequate modes reflecting the linear approach. Certain amplitude estimates with respect to time have been evaluated using Eq. (7):

$$w_i^*(x) = v_i(x)\varphi_i^* + \sum_{i \neq j} v_j(x)[\Gamma_{1ij}\varphi_j^{*3} + \mathrm{K}\,] \tag{9}$$

where it has been substituted: $w_i^*(x) = w_i(x,t^*)$, $\varphi_i^*(x) = \varphi_i(x,t^*)$ and $\dot{\varphi}_i^*(x) = 0$, t^*- time providing a maximum of $\varphi_i(t^*) = \varphi_{ic}$. The shape of modes (linear/nonlinear) is normalized in a usual way:

$$\int_0^1 w_i^2(\xi,t^*)\,d\xi = 1 \tag{10}$$

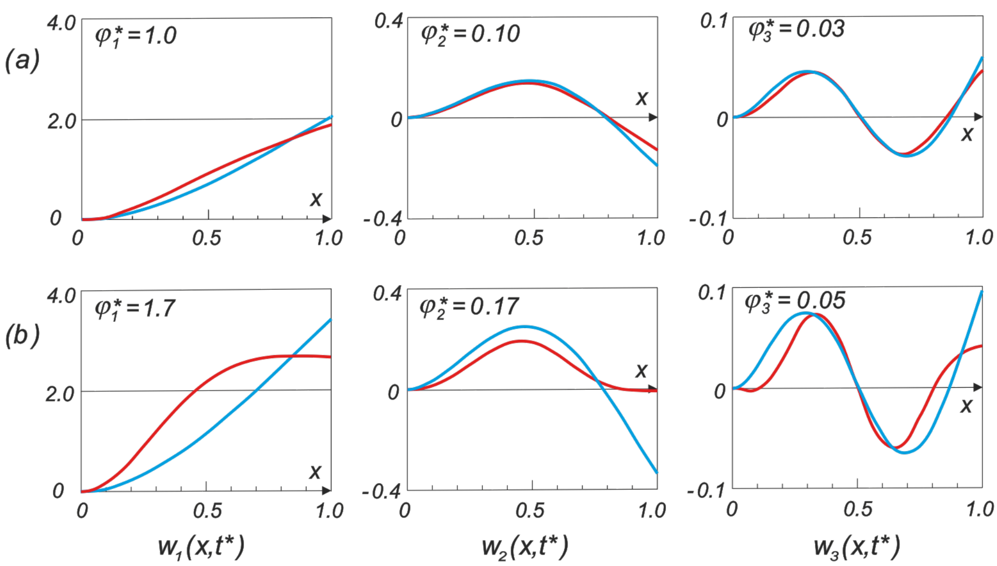

Fig. 1: Normal modes at a prismatic console: (a) linear approach - blue; (b) non-linear approach - red.

which enables to compare transparently the linear (blue) and nonlinear (red) modes for corresponding i and $\varphi_i^*(x)$ in Fig. 1. The upper triplet (a) corresponds to lower amplitude $\varphi_i^*(x)$ and the lower triplet (b) to higher amplitude $\varphi_i^*(x)$. In the latter case the differences between linear and nonlinear mode variants are higher. Differences between relevant modes increase starting zero with rising nonlinearity ratio. Because the whole solution process is approximate, there exists a critical limit φ_{ic} beyond which the convergence of the series (3) fails. Therefore it should be admitted that the above procedure works in a domain of weak nonlinearities. It is worthy to note that besides the variant (ii) - Galerkin-Petrov procedure, also the remaining variants (i) and (iii) have been processed and numerically validated. It can be concluded that especially the variants (ii) and (iii) provided very similar results of a semi-analytical investigation. Moreover, the numerical results coincided perfectly, when parameters put the system into a sub-critical domain.

4. Conclusions

Systems with continuously distributed parameters enable to construct the nonlinear normal modes. Their character and mathematical properties slightly differ from those defined for discrete systems with a couple of concentrated masses. To define the nonlinear normal modes, a discretization of the continuous system should be performed similarly like in the case of linear systems. One possibility being based at Galerkin-Petrov approach has been outlined and demonstrated at prismatic nonlinear console beam.

On the other hand, some shortcomings should also be taken into account. A system with an internal resonance between two or more modes becomes singular (remember a case of a multiple resonance of linear systems) and such modes cannot be uncoupled. Care should be taken as the system can change its character when additional energy is introduced. The internal resonance can occur due to non-linearity, despite of that under low level excitation no internal resonance exists. In such a case, a more complicated invariant manifold with more coordinates must be constructed or a center manifold formulation should be used usually together with homoclinic orbits appearance.

Finally, balancing strengths and weaknesses of NNM the advantages significantly dominate. This tool enables to concentrate analysis to the most important parts of the general response (NNM) and to consider only modes including the most of energy.

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