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FORCED VIBRATION ANALYSIS OF EULER-BERNOULLI BEAM WITH DISCONTINUITIES BY MEANS OF DISTRIBUTIONS WITHOUT DOING MODAL ANALYSIS

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Abstract: *The general equation of motion of forced vibration of Euler-Bernoulli beam has been used since it was derived by means of classical derivatives of shear force, bending moment, rotation of a cross section and deflection of the beam. However these derivatives are not defined at such points of center-line between ends of the beam in which there is a concentrated load or a concentrated support or a concentrated mass or a concentrated mass-moment of inertia or an internal hinge connecting beam segments, which are discontinuities that can be in practice. In this paper, distributional derivative for discontinuous shear force, discontinuous bending moment, and discontinuous rotation of a cross section of the beam has been applied to derive a generalized mathematical model for forced transverse vibration covering all the discontinuities mentioned. General closed-form solution to the generalized mathematical model for prismatic beam has been computed by means of symbolic programming approach via MAPLE. As a result of this new analytic approach, when computing forced steady-state response of the beam, we do not have to put together any continuity conditions at discontinuity points mentioned. The response of the beam is expressed directly without doing modal analysis.*

Keywords: Vibration, Beam, Discontinuities, Distribution, Dirac.

1. Introduction

Classical analytical method of calculating harmonic steady-state response of the beam is based on the following main steps (Rao, 2007). Firstly, we obtain a frequency equation for specific support conditions of the beam. Secondly, we solve the frequency equation for natural frequencies. Thirdly, we find orthogonal mode shapes corresponding to the natural frequencies of the beam. Finally, we express a forced response of the beam as a linear combination of the mode shapes by finding corresponding modal participation coefficients.

Applying distributional derivative for discontinuous shear force, discontinuous bending moment, and discontinuous rotation of cross section of a beam, we can derive a mathematical model for forced transverse vibration of a beam with discontinuities caused by concentrated supports or concentrated masses or concentrated mass moments of inertia or concentrated transverse forces or concentrated moments situated between ends of the beam, or hinges connecting beam segments. This mathematical model can be solved like only one differential task without dividing the beam into segments where all the continuity conditions among adjoining segments are fulfilled automatically. Using this approach, we have only four integration constants irrespective of the number of the discontinuities. Applying distributions, we do not have to compute natural frequencies, mode shapes or modal participation coefficients in analyzing forced harmonic response of beams.

2. The classical equation of motion for forced transverse vibration of Euler-Bernoulli beam

Equation of motion of a beam under distributed transverse force without discontinuities in shear force, in bending moment, in rotation of cross section or in transverse displacement of centerline of the beam is given by (Rao, 2007)

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$$\left(\frac{\partial^2}{\partial x^2}\left(E\,\mathrm{J}(x)\left(\frac{\partial^2}{\partial x^2}\mathrm{w}(x,t)\right)\right)\right)+\rho\,\mathrm{A}(x)\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right)=\mathrm{f}(x,t) \quad , \tag{1}$$

where w(x, t) is transverse displacement of the beam centerline, A(x) is cross-sectional area, J(x) is area moment of inertia, E is modulus of elasticity (Young's modulus), ρ is density, f(x,t) is the distributed transverse force.

3. A mathematical model for forced transverse vibration of Euler-Bernoulli beam with discontinuities

In order to be able to express possible discontinuities in shear force, bending moment or in rotation of cross section along a centerline of a beam mathematically without cutting the beam into segments which would be without discontinuities in support, loading or without internal hinges, distributional derivative (Schwartz, 1966; Štěpánek, 2001; Kanwal, 2004) can be used, which consists of two main parts. Its first part is a classical derivative, while the second one is distributional as a sum of products of Dirac singular distribution moved to a point of a discontinuity and a magnitude of jump discontinuity of the quantity being differentiated.

When a beam supported at concentrated supports or carrying concentrated inertia masses or subjected to concentrated transverse forces between its ends is vibrating, jump discontinuities in shear force can occur at corresponding points of centerline of the beam. By expressing the first classical partial derivative of the shear force with respect to x from the equation of motion for an element cut out of the beam, and by adding distributional parts in the form of product, Eq. (2) can be derived, where $r_i(t)$ is a reaction force at i*th* concentrated support at the point $x=a_i$ ($0<a_i< L$), L is a length of the beam, m_i is a concentrated inertia mass at the point $x=b_i$ ($0<b_i<L$), $f_i(t)$ is a concentrated transverse load at the point $x=c_i$ ($0<c_i< L$), $\delta(x-a_i)$ denotes Dirac distribution moved to the point of the discontinuity, n_1 is a number of point supports without end supports, if any, n_2 is a number of concentrated inertia masses without end masses, if any, and n_3 is a number of concentrated transverse loads without end loads, if any.

If a beam carrying concentrated masses with moments of inertia of J_i at points $x=b_i$ or subjected to concentrated moment loads $s_i(t)$ at points $x=d_i$ ($0<d_i<L$) between its ends is vibrating, jump discontinuities in bending moment can occur at these points. By expressing the first classical partial derivative of the bending moment with respect to x from the moment equation for an infinitesimal element of the beam, and by adding products of a magnitude of the jumps and Dirac distribution situated at the point of the discontinuity, Eq. (3) can be deduced, the right hand side of which is the distributional derivative of the bending moment covering n_2 plus n_4 jump discontinuities.

When a beam containing hinges connecting segments of the beam at points $x=e_i$ ($0<e_i < L$) is vibrating, jump discontinuities in rotation of the cross section of a magnitude $\psi_i(t)$ may be found at these points. By expressing the first classical partial derivative of the rotation of the cross section with the respect to x from the deformation relation of the beam centerline curvature, and by adding corresponding distributional parts, Eq. (4) can be obtained, where n_5 is a number of internal hinges, if any.

$$\begin{aligned}\frac{\partial}{\partial x}\mathrm{Q}(x,t)=\rho\,\mathrm{A}(x)\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right)+\left(\sum_{i=1}^{n_1} r_i(t)\,\delta(x-a_i)\right)+\left(\sum_{i=1}^{n_2} m_i\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right)\bigg|_{x=b_i}\delta(x-b_i)\right)\\ -\left(\sum_{i=1}^{n_3} f_i(t)\,\delta(x-c_i)\right)\end{aligned} \quad , \tag{2}$$

$$\frac{\partial}{\partial x}\mathrm{M}(x,t)=\mathrm{Q}(x,t)-\left(\sum_{i=1}^{n_2} J_i\left(\frac{\partial^2}{\partial t^2}\phi(x,t)\right)\bigg|_{x=b_i}\delta(x-b_i)\right)+\left(\sum_{i=1}^{n_4} s_i(t)\,\delta(x-d_i)\right) \quad , \tag{3}$$

$$\frac{\partial}{\partial x}\phi(x,t)=-\frac{\mathrm{M}(x,t)}{E\,\mathrm{J}(x)}+\left(\sum_{i=1}^{n_5}\psi_i(t)\,\delta(x-e_i)\right) \quad , \tag{4}$$

$$\frac{\partial}{\partial x}\mathrm{w}(x,t)=\phi(x,t) \quad . \tag{5}$$

4. Forced vibration solution

Supposing harmonic time variation of loading as

$$f_i(t) = F_i\, \mathbf{e}^{(\omega\, t\, I)} \quad , \qquad s_i(t) = S_i\, \mathbf{e}^{(\omega\, t\, I)}$$

and solution to equations (2) to (5) as

$$\mathrm{Q}(x,t) = Q_a(x)\, \mathbf{e}^{(\omega\, t\, I)} \quad , \quad \mathrm{M}(x,t) = M_a(x)\, \mathbf{e}^{(\omega\, t\, I)} \quad , \quad \phi(x,t) = \phi_a(x)\, \mathbf{e}^{(\omega\, t\, I)} \quad , \quad \mathrm{w}(x,t) = w_a(x)\, \mathbf{e}^{(\omega\, t\, I)}$$

$$r_i(t) = R_i\, \mathbf{e}^{(\omega\, t\, I)} \quad , \quad \psi_i(t) = \Psi_i\, \mathbf{e}^{(\omega\, t\, I)} \quad ,$$

where ω is circular frequency of vibration, $I^2 = -1$, and denoting amplitudes of vibration at points with concentrated inertia masses and moments of inertia as

$$\mathrm{W}_i = \lim_{x \to b_i} w_a(x) \quad , \quad \Phi_i = \lim_{x \to b_i} \phi_a(x) \quad , \tag{6}$$

a system of ordinary differential equations (7) to (10) can be derived for unknown general amplitudes of deflection, w_a , rotation of cross section, φ_a , bending moment, M_a , and shear force, Q_a , for a uniform beam as:

$$\frac{d}{dx} Q_a(x) = -\rho\, A\, w_a(x)\, \omega^2 + \left(\sum_{i=1}^{n_1} R_i\, \delta(x - a_i) \right) - \left(\sum_{i=1}^{n_2} m_i\, \mathrm{W}_i\, \omega^2\, \delta(x - b_i) \right) - \left(\sum_{i=1}^{n_3} F_i\, \delta(x - c_i) \right) \quad , \tag{7}$$

$$\frac{d}{dx} M_a(x) = Q_a(x) + \left(\sum_{i=1}^{n_2} J_i\, \Phi_i\, \omega^2\, \delta(x - b_i) \right) + \left(\sum_{i=1}^{n_4} S_i\, \delta(x - d_i) \right) \quad , \tag{8}$$

$$\frac{d}{dx} \phi_a(x) = -\frac{M_a(x)}{E\, J} + \left(\sum_{i=1}^{n_5} \Psi_i\, \delta(x - e_i) \right) \quad , \tag{9}$$

$$\frac{d}{dx} w_a(x) = \phi_a(x) \quad . \tag{10}$$

By using Laplace transform method, general solution to Eqs. (7) to (10) can be computed from which general amplitude of the deflection with integration constants in the form of initial parameters is as follows:

$$\begin{aligned}
w_a(x) = {} & \frac{(-\sinh(\beta\, x) + \sin(\beta\, x))\, \beta\, Q_a(0)}{2\, \omega^2\, m} + \frac{(\cos(\beta\, x) - \cosh(\beta\, x))\, \beta^2\, M_a(0)}{2\, \omega^2\, m} \\
& + \frac{(\sin(\beta\, x) + \sinh(\beta\, x))\, \phi_a(0)}{2\, \beta} + \left(\frac{\cos(\beta\, x)}{2} + \frac{\cosh(\beta\, x)}{2} \right) w_a(0) \\
& + \left(\sum_{i=1}^{n_4} \frac{\beta^2\, (\cos(\beta\, (x - d_i)) - \cosh(\beta\, (x - d_i)))\, S_i\, \mathrm{H}(x - d_i)}{2\, \omega^2\, m} \right) \\
& + \left(\sum_{i=1}^{n_5} \frac{\Psi_i\, \mathrm{H}(x - e_i)\, (\sin(\beta\, (x - e_i)) + \sinh(\beta\, (x - e_i)))}{2\, \beta} \right) + \left(\sum_{i=1}^{n_2} \right. \\
& (m_i\, \mathrm{W}_i\, (\sinh(\beta\, (x - b_i)) - \sin(\beta\, (x - b_i))) + (-\cosh(\beta\, (x - b_i)) + \cos(\beta\, (x - b_i)))\, \Phi_i\, J_i\, \beta)\, \beta \\
& \left. \mathrm{H}(x - b_i)/(2\, m) \right) + \left(\sum_{i=1}^{n_1} \frac{(-\sinh(\beta\, (x - a_i)) + \sin(\beta\, (x - a_i)))\, R_i\, \beta\, \mathrm{H}(x - a_i)}{2\, \omega^2\, m} \right) \\
& + \left(\sum_{i=1}^{n_3} \frac{F_i\, \mathrm{H}(x - c_i)\, (\sinh(\beta\, (x - c_i)) - \sin(\beta\, (x - c_i)))\, \beta}{2\, \omega^2\, m} \right)
\end{aligned} \quad , \tag{11}$$

where

$$m = \rho\, A \quad , \quad \omega = 2\, \pi\, f \quad , \quad \beta = \left(\frac{m\, \omega^2}{E\, J} \right)^{(1/4)} \quad . \tag{12}$$

As a simple example of using the proposed method, closed-form expression (13) for the amplitude of the deflection of forced steady-state response of a simply supported uniform beam (pinned-pinned) subjected to a concentrated harmonic force F.sin(ω.t) at $x=c$ can be expressed as follows:

$$w_a(x) = \frac{(\sin(\beta\,(x-c)) - \sinh(\beta\,(x-c)))\,\beta\,F\,\mathrm{H}(c-x)}{2\,m\,\omega^2} + \frac{\left(-\sin(\beta\,(x-c)) + \frac{\sinh(\beta\,x)\,\sinh(\beta\,(-L+c))}{\sinh(\beta\,L)} + \sinh(\beta\,(x-c)) - \frac{\sin(\beta\,x)\,\sin(\beta\,(-L+c))}{\sin(\beta\,L)}\right)\beta\,F}{2\,m\,\omega^2} \quad . \tag{13}$$

Correctness of expression (13) has been verified by evaluating classical infinite sequence (Rao, 2007; Weaver et al., 1990). A value of the expression (13) at $x=c$ has to be calculated by applying bidirectional limit because of the presence of Heaviside unit step function, i.e. H(c-x).

5. Conclusions

Contribution of this paper to harmonic forced response analysis of beams is that the mathematical model for forced transverse vibration, i.e. Eqs. (2) to (5), holds true also for discontinuous shear force, discontinuous bending moment and discontinuous rotation of cross section.

Discontinuities in shear force are supposed to be owing to idealized concentrated supports or inertia masses or concentrated transverse forces situated between ends of the beam. Likewise, discontinuities in bending moment are assumed to be due to idealized concentrated moments of inertia or concentrated moment loads situated between ends of the beam. On the contrary, discontinuities in rotation of the cross section are caused by real hinges connecting beam segments. Jump discontinuities in unknown dependently variable quantities have been expressed in corresponding distributional derivatives (2)-(4), where Dirac singular distribution, denoted as δ(x), is always moved to the point with the discontinuity mentioned, and multiplied by a magnitude of the discontinuity.

To be able to find forced response of Euler-Bernoulli beam analytically with discontinuities mentioned, Eqs. (7) to (10) have been derived for indeterminate amplitudes of the shear force, the bending moment, the rotation of the cross section and the deflection.

By using Laplace transform method, general solution to Eqs. (7) to (10) has been computed with integration constants in the form of initial parameters. A part of the general solution presented here, Eq. (11), contains H(x) as a symbol for Heaviside unit step function.

By computing limits (6), the unknown amplitudes of deflection and rotation can be expressed as functions of initial parameters. In order to determine the unknown initial parameters, four boundary conditions must be established. So as to determine the unknown reactions at concentrated supports between ends of the beam, and amplitudes of discontinuities in the rotation of the cross section at hinges connecting beam segments, corresponding deformation conditions at these points must be established. These deformation and boundary conditions create nonhomogeneous system of linear equations.

By making use of this approach, exact closed-form expressions can be found, e.g. (13), for harmonic steady-state response of uniform beams without summing infinite sequences and even without doing modal analysis.

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