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MECHANICAL RESPONSE OF COMPOSITES WITH RESPECT TO INCLUSION INTERACTION

S. Šulc*, T. Janda, J. Novák*****

Abstract: *This paper presents the major features of the μMECH micromechanical library, which gives the analytical solutions to micromechanical fields within media comprising ellipsoidal inclusions. The solutions are based on Eshelby´s stress-free eigenstrains and the equivalent inclusion method. Unlike the case of a single inclusion in an infinite matrix, for which the analytical solution is known, a fast and yet robust approach to the problem of multiple inclusions and their mutual interactions is still missing.*

Keywords: Micromechanics, Eshelby´s solution, Polynomial eigenstrains, Multiple inclusion problem.

1. Introduction

In composite materials the strain and stress concentration is of vital interest since it governs crack initializations. Numerical analyses allow us to predict material behaviour and identify regions where the concentration occurs. In the case of a composite consisting of a matrix phase and embedded heterogeneities of an ellipsoidal shape, an approach based on the solution of J. D. Eshelby (1957) to the single inclusion problem can be readily employed. We refer to an ellipsoidal heterogeneity using the term "inclusion". The present contribution describes principles of analytical methods for evaluating the mechanical fields in a particulate composite. The methods are implemented in the μMECH library.

2. Single-inclusion problem

The problem of a single ellipsoidal inclusion in an infinite matrix was successfully solved by J. D. Eshelby in 1951. Making use of Green´s function, Eshelby derived explicit formulae for the strain, stress, and displacement fluctuation fields. His fundamental contribution has been applied in many micromechanical models. In the following, we briefly review the fundamental notions related to the solution.

Assume that a material is subjected to a far field loading that would result in a constant strain $\boldsymbol{\varepsilon}^0$ if the material was homogeneous. However, due to the presence of the inclusion, strain fluctuations, denoted as $\boldsymbol{\varepsilon}^*$, occur in the material. In the sequel we use the term "perturbation" when referring to the fluctuations.

The strain perturbation together with the prescribed external load gives us the total strain as

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}^0(\mathbf{x}) + \boldsymbol{\varepsilon}^*(\mathbf{x}). \tag{1}$$

The $\boldsymbol{\varepsilon}_0$ is mapped to the equivalent eigenstrain $\boldsymbol{\varepsilon}^\tau$ through

$$\boldsymbol{\varepsilon}^\tau = \mathbf{B}\colon\boldsymbol{\varepsilon}^0 , \tag{2}$$

where the tensor $\mathbf{B}$ depends on matrix and inclusion parameters. From $\boldsymbol{\varepsilon}^\tau$ we compute the strain, stress and displacement perturbations. In the following we need only the strain perturbation $\boldsymbol{\varepsilon}^*$ computed as

* Bc. Stanislav Šulc, CTU in Prague, Faculty of Civil Engineering, Thákurova 7, 166 29, Prague, CZ, sulcstanda@seznam.cz

** Ing. Tomáš Janda, Ph.D., CTU in Prague, Faculty of Civil Engineering, Thákurova 7, 166 29, Prague, CZ, tomas.janda@fsv.cvut.cz

*** Ing. Jan Novák, Ph.D., CTU in Prague, Faculty of Civil Engineering, Thákurova 7, 166 29, Prague, CZ, novakj@cml.fsv.cvut.cz

$$\boldsymbol{\varepsilon}^{*}(\mathbf{x}) = \mathbf{D}(\mathbf{x}) : \boldsymbol{\varepsilon}^{\tau} \,. \tag{3}$$

The tensor **D** depends on the coordinates of the point of interest and again on the matrix and inclusion parameters.

3. Multiple-inclusion problem

In the case of two or more inclusions in a matrix, we have to take their mutual interaction into account. Since there is no analytical solution, we propose a self-compatibility algorithm, which iteratively corrects $\boldsymbol{\varepsilon}^{\tau}$ of each inclusion, until an acceptable tolerance η between the Euclidean norms of the two consecutive eigenstrains $\boldsymbol{\varepsilon}_i^{\tau}$ is achieved.

We approximate the effect of an inclusion *a* on an inclusion *b* as the strain perturbation caused by the inclusion *a* in the centre of inclusion *b*, which we denote $\boldsymbol{\varepsilon}_{a\to b}^{*}$. We take the perturbation as an additional external load on the inclusion *b*.

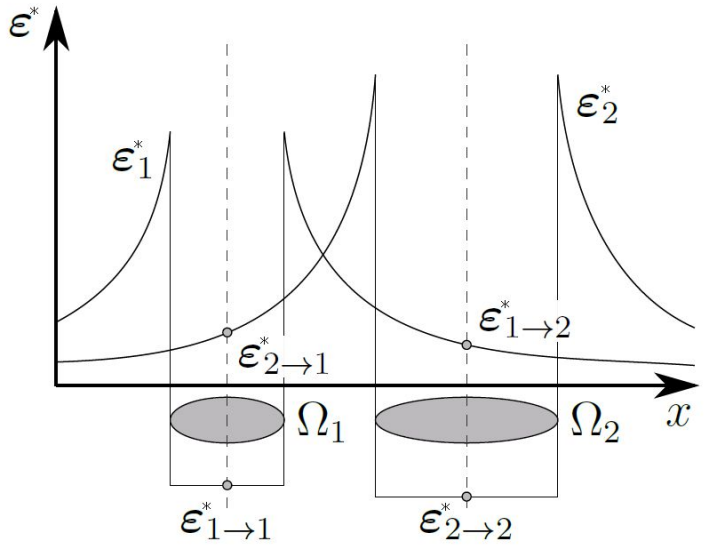

Fig. 1: Multiple-inclusion problem.

The structure of the self-compatibility algorithm for n inclusions can be summarized as follows:

1) **Do**
2) **For**(i ≤ n)
3) $\boldsymbol{\varepsilon}_i^{\tau,\mathrm{prev}} = \boldsymbol{\varepsilon}_i^{\tau}$
4) $\boldsymbol{\varepsilon}_i^{0,\mathrm{tot}} = \boldsymbol{\varepsilon}^{0} + \sum_{k\in\{1\ldots n\}\setminus\{i\}} \boldsymbol{\varepsilon}_{k\to i}^{*}$
5) $\boldsymbol{\varepsilon}_i^{\tau} = \mathbf{B}_i : \boldsymbol{\varepsilon}_i^{0,\mathrm{tot}}$
6) $\Delta\boldsymbol{\varepsilon}_i^{\tau} = \boldsymbol{\varepsilon}_i^{\tau} - \boldsymbol{\varepsilon}_i^{\tau,\mathrm{prev}}$
7) **EndFor**
8) **While**$\left(\sum_{k}^{n} \left|\left|\Delta\boldsymbol{\varepsilon}_k^{\tau}\right|\right| < \eta\right)$

At each iteration step, we update $\boldsymbol{\varepsilon}_i^{0}$ of the *i*-th inclusion as a sum of the prescribed external load $\boldsymbol{\varepsilon}^{0}$ and the strain perturbations caused by the remaining inclusions evaluated in the centre of the *i*-th inclusion. With the modified $\boldsymbol{\varepsilon}_i^{0}$ we update the equivalent eigenstrain $\boldsymbol{\varepsilon}_i^{\tau}$. The comparison of results with and without using this algorithm are in Fig. 3.

5. Polynomial eigenstrains

Introducing the effect of the remaining inclusions only through strain perturbation at centre of the treated inclusion gives sufficiently good results if the inclusions are distant enough such that the gradient of the eigenstrain is negligible compared to the average value of the eigenstrain in this case. However, when the inclusions get closer, the gradient starts to play an important role, see Fig. 1. In the vicinity of the considered inclusion it is not even close to linear function. As a result, the average value for an inclusion deviates significantly from the value in its centre leading to the underestimated results.

This drawback can be diminished by replacing the constant eigenstrains with a polynomial function and using the known analytical solutions for polynomial eigenstrains. In particular, we have implemented the linear approximation, which seems to be sufficiently accurate even for short distance interaction.

As an example we have analyzed a 2D problem consisting of three circular inclusions. Geometry of the task is depicted in Fig. 2. All the inclusions have radius 1.0 m. Young´s modulus of the inclusion material is 10.0 with the Poisson´s ratio of value 0.3. The matrix Young´s modulus was assumed 1.0 and the Poisson´s ratio 0.2. The prescribed external load corresponded to the uniaxial strain with the only non-zero component $\varepsilon_x^0 = 1.0$.

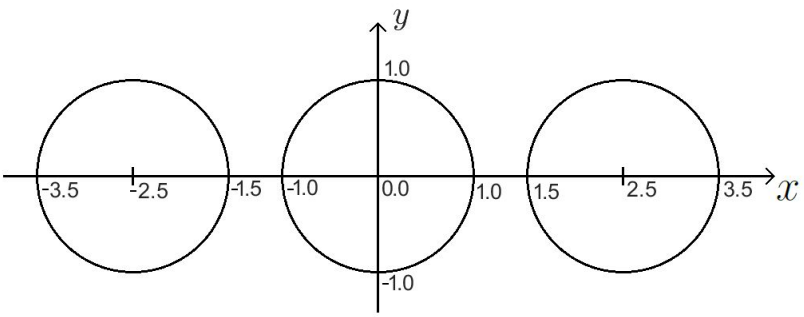

Fig.2: Scheme of the three-inclusion problem.

Fig. 3 shows a comparison of the proposed solutions and the reference solution obtained with Finite Element Method (FEM). Outside the group of inclusions the results provided by the polynomial solution closely resemble the FEM results. The importance of the self-compatibility algorithm and polynomial approximation of $\boldsymbol{\varepsilon}^{\tau}$ gets further pronounced in the regions among inclusion, where the interactions play a significant role.

4. Internal perturbation fields

In the multiple-inclusion problem the internal strain and stress perturbations are not constant and we cannot simply sum all the perturbations as we do in the case of points that are not in any inclusion. The first option to compute the internal fields is to use the polynomial solution also for the internal points. However, this leads to the loss of detailed distribution of eigenstrain which is limited by the polynomial approximation.

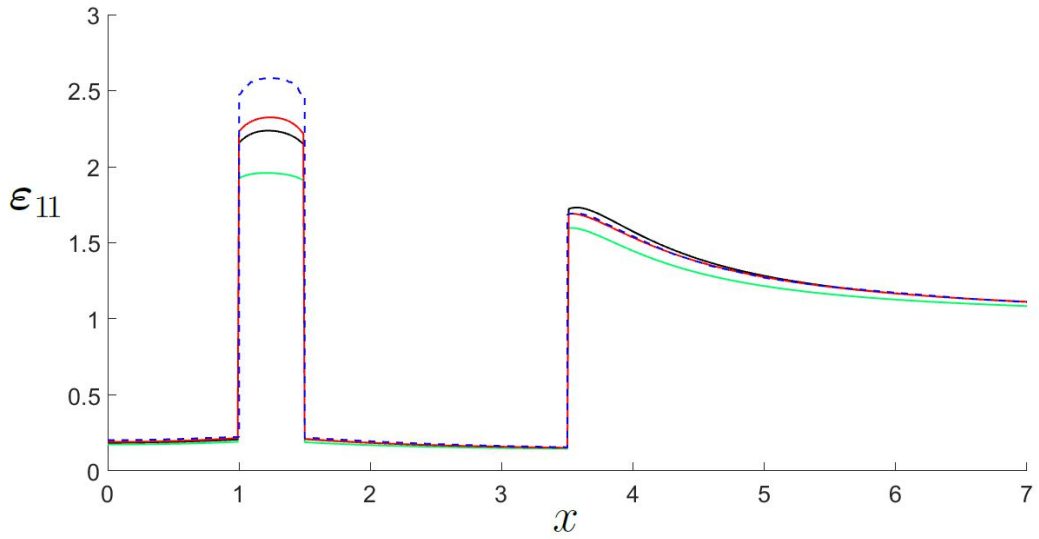

Fig.3: Results of the three-inclusion problem. Blue line: FEM.
Red line: Linear approximation of the $\boldsymbol{\varepsilon}^{\tau}$ with self-compatibility algorithm.
Black line: Constant $\boldsymbol{\varepsilon}^{\tau}$ with self-compatibility algorithm.
Green line: Constant $\boldsymbol{\varepsilon}^{\tau}$ without self-compatibility algorithm

We get the most accurate results employing the same principle as in the self-compatibility algorithm. We compute the strain perturbations of all other inclusions first. We sum them with the prescribed external load and take it as a new load for the inclusion with the point inside. Then, we map this load to $\boldsymbol{\varepsilon}^{\tau}$ and compute $\boldsymbol{\varepsilon}^{*}$ from Eq. 3.

We compute the stress and displacement perturbations in the usual way, the only difference is that we use this recalculated $\boldsymbol{\varepsilon}^{\tau}$ instead of the one evaluated in the self-compatibility algorithm.

6. Conclusions

The present results confirm the importance of accounting for the interaction among inclusions. We have presented two approximate solutions. In the first, the interaction was taken into account by employing the self-compatibility algorithm arising from the assumption of the constant equivalent eigenstrain within inclusions. In the second approach, we have extended this approach with the linear approximation. As expected, the latter approach gives better results when compared with the reference solution obtained with a Finite Element Method (FEM). Albeit the accuracy of the FEM solution was not reached the present methods yielded sufficiently accurate results with respect to their intended application. The µMECH library can solve much larger problems in a fraction of time when compared to FEM and thus provide fluctuation fields as a global enrichment functions for the generalized finite element methods. Our further effort aims improving the accuracy by quadratic polynomial approximation of the eigenstrains.

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