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# INFLUENCE OF AMPLITUDE ON FREE VIBRATION FREQUENCY OF A PARTIALLY TENSIONED COLUMN

**S. Uzny*, K. Sokół**, M. Osadnik*****

**Abstract:** *The non-linear vibrations of a slender system subjected to an external force applied between the elements of a structure are discussed in this paper. The boundary problem has been formulated on the basis of Hamilton's principle and the small parameter method. Taking into account the solution of the boundary problem, the parameter describing the relation between the amplitude and natural vibration frequency has been obtained. In the presented investigations, only the first vibration frequency is discussed. The numerical simulations of the vibration frequencies are done for variable location of the external load.*

**Keywords: Non-linear column, natural vibration, amplitude, instability.**

## 1. Introduction

The behaviour of certain structures can only be described using non-linear differential equations. This particularly holds for structures where vibrations have to be taken into account. Such an approach has been discussed by many authors in recent years. Various methods of solving non-linear boundary problems can be found in the literature. In the work by Szemplińska-Stupnicka (1983) a continuous non-linear system in the form of a beam was investigated. The author used the Ritz method in the solution to the problem. Examples of the relationship between the amplitude and the vibration frequency were presented. Nonlinear vibrations of a beam with an elastic axial restraint were discussed by Prathap (1978). The author investigated how the amplitude affects the vibration frequency for different slendernesses of the structure as well as the influence of different stiffening of the pinned support in the longitudinal direction. Awrejcewicz et.al. (2011) studied a flexible nonlinear Euler-Bernoulli-type beam for different boundary conditions. The continuous boundary problem was reduced to a finite-dimensional one and solved using the Runge-Kutta method. Sokół (2014) studied the influence of a crack present in one of rods of a multi-member structure on the amplitude – vibration frequency relationship. Tomski and Kukla (1989) performed theoretical and numerical studies of a column subjected to an eccentrically applied Euler's load on both ends of the system. Additionally, they used a translational spring in order to reduce the longitudinal displacement on one of the ends. In the solution to the boundary problem the small parameter method was used. As a result the parameters describing the relation between the linear and non-linear vibration frequency and longitudinal (linear/non-linear) force for different amplitudes and spring stiffness were obtained. Structures in the form of coaxial tubes or flat frames composed of elements with different rigidity are also geometrically non-linear and were presented in papers (Tomski et al. 2007, Uzny 2011). These types of structures are characterized by two forms of equilibrium: rectilinear and curvilinear.

* Sebastian Uzny, Prof.: Institute of Mechanics and Machine Design Foundation, Częstochowa University of Technology Dąbrowskiego 73, 42-200 Częstochowa, Poland, uzny@imipkm.pcz.pl

** Krzysztof Sokół, PhD.: Institute of Mechanics and Machine Design Foundation, Częstochowa University of Technology Dąbrowskiego 73, 42-200 Częstochowa, Poland, sokol@imipkm.pcz.pl

*** Michał Osadnik, Eng.: Institute of Mechanics and Machine Design Foundation, Częstochowa University of Technology Dąbrowskiego 73, 42-200 Częstochowa, Poland, osadnik@imipkm.pcz.pl

The main scope of this paper is to estimate the differences between the linear and non-linear vibration frequency of a system subjected to axial load for a given magnitude of the parameter associated with the amplitude of natural vibrations. The proposed system can be modelled by a screw with a nut moving along its length. Additionally, in the real system, the nut is subjected to an external load.

## 2. Formulation of the boundary problem

The slender system discussed in this paper is presented in the Figure 1. It is constructed as a fixed – fixed rod loaded by means of force *P*, which is applied axially between the elements of the structure. The point of the force placement is described by parameter $\zeta$ ($\zeta = l_1/l$). The whole structure is characterized by constant bending stiffness ($(EJ)_1 = (EJ)_2 = (EJ)$), compression stiffness ($(EA)_1 = (EA)_2 = (EA)$) and mass ($(\rho A)_1 = (\rho A)_2 = (\rho A)$) (where: $E_i$ – Young's modulus, $\rho_I$ – density, $A_i$ – cross sectional area, $J_i$ – geometrical axial moment of inertia of the cross section of *i*-th element of the structure).

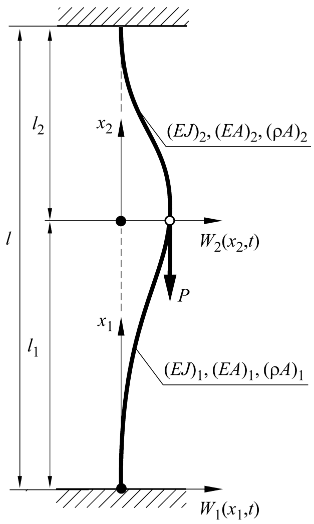


*Figure 1. An investigated system*

The differential equation of motion in transversal and longitudinal directions are as follows:

$$\frac{\partial^4 w_i(\xi_i,\tau)}{\partial \xi_i^4} + k_i^2(\tau)\frac{\partial^2 w_i(\xi_i,\tau)}{\partial \xi_i^2} + \Omega_i^2 \frac{\partial^2 w_i(\xi_i,\tau)}{\partial \tau^2} = 0 \qquad (1)$$

$$u_i(\xi_i,\tau) - u_i(0,\tau) = -\frac{k_i^2(\tau)}{\theta_i}\xi_i - \frac{1}{2}\int_0^{\xi_i}\left(\frac{\partial w_i(\xi_i)}{\xi_i}\right)^2 d\xi_i \qquad (2)$$

Equations (1) and (2) are written in the non-dimensional form by means of the relationships:

$$\xi_i = \frac{x_i}{l_i},\ w_i(\xi_i,\tau) = \frac{W_i(x_i,\tau)}{l_i},\ u_i(\xi_i,\tau) = \frac{U_i(x_i,\tau)}{l_i},\ k_i^2(\tau) = \frac{S_i(\tau) l_i^2}{(EJ)_i},$$

$$\Omega_i^2 = \frac{(\rho A)_i \omega^2 l_i^4}{(EJ)_i},\ \tau = \omega t,\ \Theta_i = \frac{A_i l_i^2}{J_i},\ i = 1, 2. \qquad (3a\text{-}g)$$

Where: $S_i(\tau)$ – internal force in the *i*-th element of the structure, $\omega$ - vibration frequency, $W_i(x_i,\tau)$, $U_i(x_i,\tau)$ – transversal and longitudinal displacements, respectively.

The boundary conditions (in the non-dimensional form) are as follows:

$$u_1(0,\tau) = u_2(1.\tau) = w_1(0,\tau) = \left.\frac{\partial w_1(\xi_1,\tau)}{\partial \xi_1}\right|_{\xi_1=0} = w_2(1,\tau) = \left.\frac{\partial w_2(\xi_2,\tau)}{\partial \xi_2}\right|^{\xi_2=1} = 0\ ;\ u_1(1,\tau) = u_2(0.\tau)$$

$$u_1(0,\tau)=u_2(1.\tau)=w_1(0,\tau)=\left.\frac{\partial w_1(\xi_1,\tau)}{\partial \xi_1}\right|_{\xi_1=0}=w_2(1,\tau)=\left.\frac{\partial w_2(\xi_2,\tau)}{\partial \xi_2}\right|^{\xi_2=1}=0\,;\; w_1(1,\tau)=w_2(0,\tau)$$

$$\frac{(EJ)_1}{l_1^2}\left.\frac{\partial^3 w_1(\xi_1,\tau)}{\partial \xi_1^3}\right|^{\xi_1=1}-\frac{(EJ)_2}{l_2^2}\left.\frac{\partial^3 w_2(\xi_2,\tau)}{\partial \xi_2^3}\right|_{\xi_2=0}+P\left.\frac{\partial w_2(\xi_2,\tau)}{\partial \xi_2}\right|_{\xi_2=0}=0\,;\; S_1-S_2=P$$

$$\frac{(EJ)_1}{l_1}\left.\frac{\partial^2 w_1(\xi_1,\tau)}{\partial \xi_1^2}\right|^{\xi_1=1}-\frac{(EJ)_2}{l_2}\left.\frac{\partial^2 w_2(\xi_2,\tau)}{\partial \xi_2^2}\right|_{\xi_2=0}=0\,;\;\left.\frac{\partial w_1(\xi_1,\tau)}{\partial \xi_1}\right|^{\xi_1=1}=\left.\frac{\partial w_2(\xi_2,\tau)}{\partial \xi_2}\right|_{\xi_2=0} \tag{4a-r}$$

In the final formulation of the boundary problem the small parameter method is used due to the non-linearity in equation (2). All the non-linear elements are written in the power series of the small parameter. In this paper only the rectilinear form of static equilibrium is discussed, for which, the series are as follows:

$$w_i(\xi,\tau)=\sum_{j=1}^{N}\varepsilon^{2j-1}w_{i2j-1}(\xi,\tau)+O\left(\varepsilon^{2(N+1)}\right),\; u_i(\xi,\tau)=u_{i0}(\xi)+\sum_{j=1}^{N}\varepsilon^{2j}u_{i2j}(\xi,\tau)+O\left(\varepsilon^{2(N+1)}\right) \tag{5a}$$

$$k_i^2(\tau)=k_{i0}^2+\sum_{j=1}^{N}\varepsilon^{2j}k_{i2j}^2(\tau)+O\left(\varepsilon^{2(N+1)}\right),\; \Omega_i^2=\Omega_{i0}^2+\sum_{j=1}^{N}\varepsilon^{2j}\Omega_{i2j}^2+O\left(\varepsilon^{2(N+1)}\right) \tag{5b}$$

The above equations are solved sequentially for: the estimation of the distribution of the external force onto elements of the structure, linear vibration frequency $\omega_0$, internal forces induced by vibrations, non-linear vibration frequency $\omega_2$.

## 3. The results of numerical simulations

The discussion of the results is done with the use of the non-dimensional parameters:

$$\lambda=\frac{Pl^2}{EJ}\,,\; \zeta_\omega=\frac{\omega-\omega_0}{\omega_0}100\%\,,\text{ where: } \omega=\sqrt{\omega_0^2+\varepsilon^2\omega_2^2} \tag{6a-c}$$

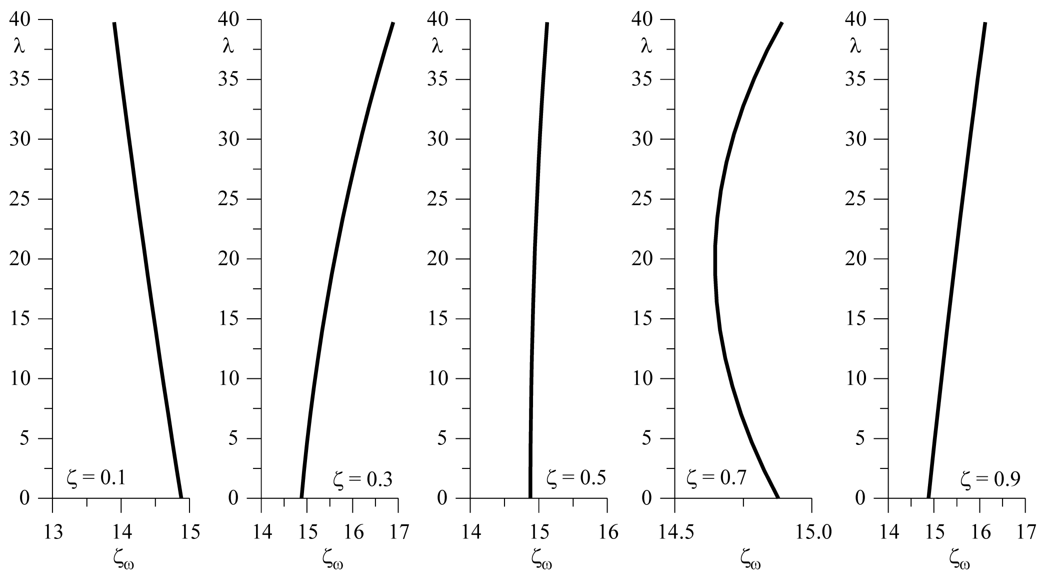


*Figure 2. The change of $\zeta_\omega$ parameter in relation to external load $\lambda$*

The numerical investigations are done at external load level $\lambda \in \langle 0,25\rangle$ and at amplitude of vibration parameter $\varepsilon = 0.008$. The proposed magnitude of the small parameter $\varepsilon = 0.008$ refers to an amplitude of vibration approximately equal to double the magnitude of the minimum radius of inertia of the cross section of the column. In Figure 2, the change in parameter $\zeta_\omega$ in relation to the location point of external load $\zeta$ ($\zeta$ = 0.1; 0.3; 0.5; 0.7; 0.9) is presented. In this study only an influence of an amplitude on the first vibration frequency is presented and discussed.

The degree of the influence of the amplitude on the vibration frequency of the system depends on the external load (parameter $\lambda$) and the point of its location (parameter $\zeta$). In the investigated range of external load ($\lambda \in \langle 0,40 \rangle$), at amplitude corresponding to small parameter $\varepsilon$ = 0.008 the smallest differences in the plotted curves $\lambda(\zeta_\omega)$ can be found when the external force is located in the middle of the structure ($\zeta$ = 0.5). For a smaller magnitude of $\zeta$ ($\zeta$ = 0.1) it can be seen that the differences between the frequency with non-linear part $\omega$ (where: $\omega^2 = \omega_0^2 + \varepsilon^2\omega_2^2$; $\omega_0$ – linear component independent from amplitude of vibration, $\omega_2$ – non-linear component dependent on amplitude of vibration) and basic $\omega_0$ become smaller as the external load magnitude increases. In the case where $\zeta$ = 0.7, the considered difference between $\omega$ and $\omega_0$ is smaller for the small magnitude of the external load than in the case when the external load is increasing.

## 4. Conclusions

The numerical studies presented in this paper concern the estimation of the influence of the amplitude of vibration on the natural vibration frequency of slender system which is partially tensioned in a rectilinear form of static equilibrium. It has been shown that the size of the influence of the amplitude on the first vibration frequency greatly depends on the location point of the external load as well as on its magnitude. On the basis of the results of numerical simulations it can be stated that the point which is the least sensitive to the placement of the external load is the one located in the middle to the total length of the column (see curves external load ($\lambda$) – frequency difference ($\zeta_\omega$) between $\omega$ and $\omega_0$). In the investigated external load range and its location, parameter $\zeta_\omega$ varies from ~14% up to ~17%. In the future it is planned to extend the studies presented in this paper to include the curvilinear form of static equilibrium and discreet elements which can affect the natural vibration frequency.

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