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STEADY FLOWS OF SECOND-GRADE FLUIDS SUBJECT TO STICK-SLIP BOUNDARY CONDITIONS

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Abstract: *This paper is devoted to analyzing steady flows of second-grade fluids in a plane channel with impermeable walls under the assumption that the flows are driven by constant pressure gradient. We use various boundary conditions, namely, no-slip, stick-slip, free-slip, and mixed boundary conditions, assuming that the solid walls may include two parts differing by their physical properties. For each of the considered boundary value problems, we present the exact solutions, which characterize the velocity field and the pressure in the channel. Moreover, for model parameters we establish explicit relationships that guarantee the slip (no-slip) regime on the solid walls.*

Keywords: Non-Newtonian fluids, Second-grade Fluids, Plane channel flows, Stick-slip boundary conditions, Exact solutions.

1. Introduction

Many of the materials used in practice belong to the class of fluids of complexity N (Rivlin, 1955 and Dunn, 1995). For such fluids, the Cauchy stress tensor $\mathbf{T}$ is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{F}(\mathbf{A}_1, ..., \mathbf{A}_N),$$

where p is the pressure, $\mathbf{F}$ is a frame indifferent response function, and $\mathbf{A}_1$, $\mathbf{A}_2$, ..., $\mathbf{A}_N$ are the first N Rivlin–Ericksen tensors:

$$\begin{aligned} \mathbf{A}_1 &= \nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}, \\ \mathbf{A}_j &= D_t\mathbf{A}_{j-1} + \mathbf{A}_{j-1}\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_{j-1} \end{aligned}$$

for $j = 2, 3, ..., N$. Here, $\mathbf{v}$ is the velocity and D_t is the material time derivative.

If $\mathbf{F}$ is a polynomial of degree N, then the appropriate fluid is called a *fluid of grade N*.

The classical incompressible Newtonian fluid $\mathbf{T} = -p\mathbf{I} + \nu\mathbf{A}_1$ is a fluid of grade 1. Well-studied nonlinear-viscous fluids, for which the constitutive equation is given by fluid $\mathbf{T} = -p\mathbf{I} + \nu(\mathbf{A}_1)\mathbf{A}_1$, are belong to the class of fluids of complexity 1.

In this work, we deal with the second-grade fluids

$$\mathbf{T} = -p\mathbf{I} + \nu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where ν is the viscosity coefficient, α_1 and α_2 are the normal stress moduli. It was shown in the paper Dunn (1974) that $\nu \geq 0$, $\alpha_1 \geq 0$, and $\alpha_1 + \alpha_2 = 0$. Using the notation $\alpha = \alpha_1 = -\alpha_2$, we rewrite (1) as follows

$$\mathbf{T} = -p\mathbf{I} + \nu\mathbf{A}_1 + \alpha\mathbf{A}_2 - \alpha\mathbf{A}_1^2. \tag{2}$$

We are interested in finding exact solutions for steady flows of the fluid (2) in a plane infinite channel. The main feature of this paper is that the different types of boundary conditions on solid walls are used. In addition to the classical no-slip condition $\mathbf{v} = \mathbf{0}$, we consider stick-slip, free-slip, and mixed boundary conditions, assuming that the channel walls may differ in their physical properties. Such models are

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interesting both in theoretical research and in dealing with applied problems. The importance of wall slip effect and its impact on different characteristics of fluid flows, especially in the case of non-Newtonian fluids, is noted in many studies (see, e.g., the recent papers Rajagopal (2003), Kazatchkov (2010), Hatzikiriakos (2012) and the references cited therein). Significant contribution to the study of the causes and mechanisms of slipping of fluids and disperse systems on solid surfaces was made by Tolstoy (1953).

In this paper for each of the considered boundary value problems we present exact solutions, which determine the velocity field and the pressure in the flow region. Moreover, for model parameters we establish explicit relationships that guarantee the slip (no-slip) regime on the solid walls of the channel.

It should be mentioned that Hron et al. (2008) investigate steady flows of fluids of complexity 2 in a plane channel and a cylindrical pipe and flows between two rotating concentric cylinders subject to the Navier slip boundary condition, which is a special case (or more precisely, a limit case) of the stick-slip condition. Existence, uniqueness and qualitative properties of the solutions to motion equations of second-grade fluids in arbitrary bounded domains with Navier's slip boundary condition are studied in the papers Le Roux (1999), Busuioc (2003), Tani (2005), and Baranovskii (2015).

2. Statement of the problems

It is well known that the steady motion of a homogeneous incompressible fluid is governed by the following system of equations:

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \operatorname{div}\mathbf{S} - \nabla p + \rho\mathbf{g}, \quad \operatorname{div}\mathbf{v} = 0, \tag{3}$$

where ρ is the density of the fluid, **v** is the velocity, **S** is the deviator of the stress tensor **T**, p is the pressure, and **g** is the body force per unit mass. The operators div and ∇ are the divergence and the gradient, respectively, with respect to the space variables x, y, z.

Let us consider fluid flows between (infinite) parallel plates $z = -h$ and $z = h$. We assume that the flows are driven by constant pressure gradient

$$\frac{\partial p}{\partial x} = -\xi, \quad \xi > 0, \tag{4}$$

and $\mathbf{g}^{\mathrm{T}} = (0,0,-g)$. Thus, we deal with the plane Poiseuille flow. For such flow, we obviously have $v_1 = u(z)$, $v_2 = 0$, $v_3 = 0$, where $u = u(z)$ is a function. This yields that $(\mathbf{v}\cdot\nabla)\,\mathbf{v}$ and $\operatorname{div}\mathbf{v} = 0$. Hence system (3) is reduced to

$$\operatorname{div}\mathbf{S} = \nabla p - \rho\mathbf{g}. \tag{5}$$

Assume that the fluid obeys the constitutive relation (2). Taking into account $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$, we can rewrite (5) in the form

$$\operatorname{div}(\nu\mathbf{A}_1 + \alpha\mathbf{A}_2 - \alpha\mathbf{A}_1^2) = \nabla p - \rho\mathbf{g}. \tag{6}$$

We shall use this equation for handling second-grade fluid flows in the channel $-h \le z \le h$. Note that the unknowns of (6) are u and p.

Of course, (6) must be supplemented with appropriate boundary conditions in order to take into account physically important solutions. Experimental data and theoretical works point to various possibilities for the behaviour of fluid flows on solid walls, see for example the paper Rajagopal (2003).

In this paper, we examine the three boundary value problems describing flows of second-grade fluids in the plane channel with impermeable walls.

Problem (A) *Find functions u and p that satisfy* (4), (6) *subject to the no-slip conditions* $\mathbf{v} = \mathbf{0}$ *on* $z = \pm h$.

Problem (B) *Find functions u and p that satisfy* (4), (6) *subject to the impermeability condition* $\mathbf{v}\cdot\mathbf{n} = 0$ *and the stick-slip boundary condition on* $z = \pm h$:

$$\|(\mathbf{Tn})_{\tan}\|_{\mathbf{R}^3} \le \sigma \quad \Rightarrow \quad \mathbf{v}_{\tan} = \mathbf{0},$$

$$\|(\mathbf{Tn})_{\tan}\|_{\mathbf{R}^3} > \sigma \quad \Rightarrow \quad (\mathbf{Tn})_{\tan} = -(\sigma + k\,\|\mathbf{v}_{\tan}\|_{\mathbf{R}^3})\frac{\mathbf{v}_{\tan}}{\|\mathbf{v}_{\tan}\|_{\mathbf{R}^3}}.$$

In this formulas, **n** is the exterior unit normal vector on the channel walls, $\mathbf{v}_{\tan}$ denotes the tangential component of **v**, $k > 0$ is the slip coefficient, and $\sigma \geq 0$ is the slip threshold.

Problem (C) *Find functions u and p that satisfy* (4), (6) *subject to the impermeability condition* $\mathbf{v} \cdot \mathbf{n} = 0$ *on* $z = \pm h$ *and the mixed boundary conditions*:

$$||(\mathbf{Tn})_{\tan}||_{\mathbf{R}^3} \leq \sigma \quad \Rightarrow \quad \mathbf{v}_{\tan} = \mathbf{0}, \quad \text{for } z = h,$$

$$||(\mathbf{Tn})_{\tan}||_{\mathbf{R}^3} > \sigma \;\; \Rightarrow \;\; (\mathbf{Tn})_{\tan} = -(\sigma + k_1 ||\mathbf{v}_{\tan}||_{\mathbf{R}^3}) \frac{\mathbf{v}_{\tan}}{||\mathbf{v}_{\tan}||_{\mathbf{R}^3}}, \quad \text{for } z = h,$$

$$(\mathbf{Tn})_{\tan} = -k_2 \mathbf{v}_{\tan} \;\; \text{on} \;\; z = -h,$$

where $k_1 > 0$ *and* $k_2 \geq 0$.

Remark. The last equality is the free-slip boundary condition, also known as Navier's slip law.

3. Results

Our main results are given in the following tables.

Tab. 1: Overview of solution to problem (A).

Velocity	$u(z) = -\frac{\xi}{2\nu}(z^2 - h^2)$
Pressure	$p(x,z) = -\xi x + \alpha(u'(z))^2 + \rho g(h - z)$

Tab. 2: Overview of solution to problem (B).

	The case $\xi h \leq \sigma$	The case $\xi h > \sigma$
Velosity	$u(z) = -\frac{\xi}{2\nu}(z^2 - h^2)$	$u(z) = -\frac{\xi}{2\nu}(z^2 - h^2) + \frac{\xi h - \sigma}{k}$
Pressure	$p(x,z) = -\xi x + \alpha(u'(z))^2 + \rho g(h - z)$	$p(x,z) = -\xi x + \alpha(u'(z))^2 + \rho g(h - z)$
Regime on the channel walls	*the no-slip regime*	*the slip regime*

Tab. 3: Overview of solution to problem (C).

	The case $\xi h \leq \sigma\left(1 - \frac{\nu}{2(\nu + k_2 h)}\right)$	The case $\xi h > \sigma\left(1 - \frac{\nu}{2(\nu + k_2 h)}\right)$
Velosity	$u(z) = -\frac{\xi}{2\nu}(z^2 - h^2) - \frac{\xi h}{\nu + 2k_2 h}(z - h)$	$u(z) = -\frac{\xi}{2\nu}(z^2 - h^2) - \frac{\xi h(k_1 - k_2) + \sigma k_2}{2hk_1k_2 + \nu(k_1 + k_2)} z + \frac{\xi h^2(k_1 + k_2) + 2\xi h\nu - \sigma hk_2 - \nu\sigma}{2hk_1k_2 + \nu(k_1 + k_2)}$
Pressure	$p(x,z) = -\xi x + \alpha(u'(z))^2 + \rho g(h - z)$	$p(x,z) = -\xi x + \alpha(u'(z))^2 + \rho g(h - z)$
Regime on the wall $z = -h$	*the slip regime*	*the slip regime*
Regime on the wall $z = h$	*the no-slip regime*	*the slip regime*

4. Conclusions

In this note we present exact solutions of some boundary value problems describing the steady flows of second-order fluids in a plane channel under the no-slip boundary condition as well as stick-slip and mixed boundary conditions.

These solutions show that the pressure in the channel significantly depends on the normal stress coefficient α, especially in those subdomains, where the change of flow velocity is large (in the transverse direction of the channel). At the same time, the velocity field is independent of α, and therefore coincides with the velocity field that occurs in the case of a Newtonian fluid ($\alpha = 0$).

In a description of stick-slip flows, the key point is value of ξh, where ξ is module of the gradient pressure, h is the half-channel height. If ξh exceeds some threshold value, then the slip regime holds at solid surfaces, otherwise the fluid adheres to the walls of the channel.

If it is assumed that on one part of the boundary the free-slip condition (Navier's condition) is provided, while on the other one the stick-slip condition holds, then for the slip regime the corresponding threshold value is reduced to a certain extent, but not more than twice.

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