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APPLICATION OF FRACTIONAL CALCULUS IN HARMONIC OSCILATOR

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Abstract: *This paper presents two models of the vibrating system derived from the classical equation of harmonic oscillator, which were reduced to a single model. For the description of the reduced model one used the Fractional Calculus and the concept of derivative-integral defined by Caputo.*

Keywords: Harmonic oscillator, Caputo derivative, fractional calculation, Kelvin – Voight model.

1. Introduction

The developing industry and new technologies of components and systems manufacture led to the development of new mathematical tools for describing dynamic processes occurring in these systems or components. Such a new mathematical tool that caught the attention of scientists in recent years is the Fractional Calculus. The use of fractional calculus consists in the generalization of the classic derivatives (and integrals which are directly connected to derivatives) in the real numbers. The basic concept of this calculus is the notion of derivative-integral of fractional order. The most popular and most commonly used (also in this paper) is the concept of derivative - integral defined by Caputo. Although the derivatives of non-total order and differential equations of fractional order are known in the mathematical areas (Blasiak, S., 2016), recently they are used in the field of mechanics and physics. Differential equations of fractional order are used among others to describe physical phenomena occurring in electromagnetism, propagation of energy, thermal stresses (Blasiak, S. et al., 2014), relaxation vibrations (Blasiak, M., 2016; Blasiak, S. and Zahorulko, 2016; Nowakowski et al., 2016), viscoelasticity, thermoelasticity and pneumatic systems (Laski et al., 2014; Takosoglu et al., 2016; Zwierzchowski, 2016). Compared to the classical calculus, the main advantage of fractional calculation is the possibility of investigation of the nonlocal response of mechanical systems. In the papers (Stanislavsky, 2005; Yonggang and Xiu'e, 2010) the authors defined the mechanical oscillators replacing the total derivatives with derivatives of fractional order. These equations were solved numerically and analytically using Laplace transform. In (Kaczorek, 2013) the author by using the Caputo derivative presented analytical solution of the fractional damped oscillator equation.

Harmonic oscillators appear in different areas of classical mechanics, electronics, experimental physics and quantum physics. Harmonic oscillators are defined by ordinary differential equations.

The following paper presents two models of the vibrating system derived from the classical equation of harmonic oscillator, which was reduced to a single model described by differential equation of fractional order.

2. Methods

Classical damped harmonic oscillator has three main components: the mass m (kg), the damping factor c (Ns/m), rigidity factor k (N/m):

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$$m\frac{d^2x(t)}{dt^2}+c\frac{dx(t)}{dt}+k\,x(t)=v(t) \tag{1}$$

for extortion $v(t)=A_0\cdot\sin(\omega t)$

Two cases were concerned:

The first model for *m=0* and the following initial condition: x(0)=0

$$c\frac{dx(t)}{dt}+k\,x(t)=v(t) \tag{2}$$

Second model for *c=0* and the following initial conditions: x'(0)=0, x(0)=0

$$m\frac{d^2x(t)}{dt^2}+k\,x(t)=v(t) \tag{3}$$

Caputo's fractional derivative of a function $f(t)$ is defined as:

$$\frac{d^\alpha}{dt^\alpha}f(t)\equiv {}_0^C D_t^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n f(\tau)}{d\tau^n}d\tau \\ t>0,\ n-1<\alpha<n \tag{4}$$

In this definition, both the derivative and integral are defined on the interval, rather than at a point, as seen from the classical definition of a derivative.

Taking into account the relations of (4) to (1) and (2) equations and introducing replacement coefficients *a* and *b*, one brought them to a fractional equation of general form

$$a\,{}_0^C D_t^\alpha x(t)+b\,x(t)=v(t) \tag{5}$$

By applying the Laplace transform to (5) equation, one received:

$$g(s)=\frac{\bar{v}(s)}{as^\alpha+b}=\frac{1}{a}\frac{1}{s^\alpha+\frac{b}{a}}\cdot\bar{v}(s) \tag{6}$$

After applying the inverse Laplace transform in (6) equation, one obtained:

$$G(t)=\frac{1}{a}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{b}{a}t^\alpha\right) \tag{7}$$

where $E_{\alpha,\alpha}(t)$ is the Mittag-Leffler function, defined as follows:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)} \qquad z\in C,\quad \beta\in C,\quad \alpha>0,\ \beta>0 \tag{8}$$

In case when $\alpha=\beta=1$, ML function has the following form $E_{1,1}(z)=e^z$ and for $\alpha=\beta=2$:

$$E_{2,2}(z)=\cosh\left(\sqrt{z}\right).$$

Using the general dependence on the inverse Laplace transform, and the Green's function, one wrote

$$x(t)=\mathscr{L}^{-1}\left\{\frac{1}{s^\alpha+\omega^2}\cdot\bar{f}(s)\right\}=\int_0^t G(t-\tau)\,f(\tau)\,d\tau \tag{9}$$

and received the equation describing the oscillations of the system:

$$x(t)=\frac{1}{a}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{b}{a}(t-\tau)^\alpha\right)v(\tau)\,d\tau \tag{10}$$

The results of numerical analysis of the current model are presented later in this article.

3. Results

For calculations one adopted the values of substitute coefficients *a* = *b* = *1* in equation (5).

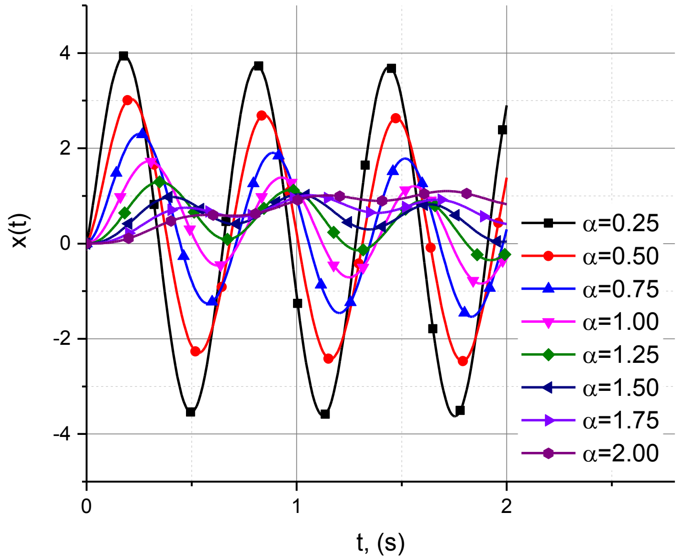

Fig. 1: Progression of vibrations of the system for $0 < \alpha \leq 2$.

The charts show the analysis of the course of vibrations for different values of α describing the order of equation (5) of fractional order.

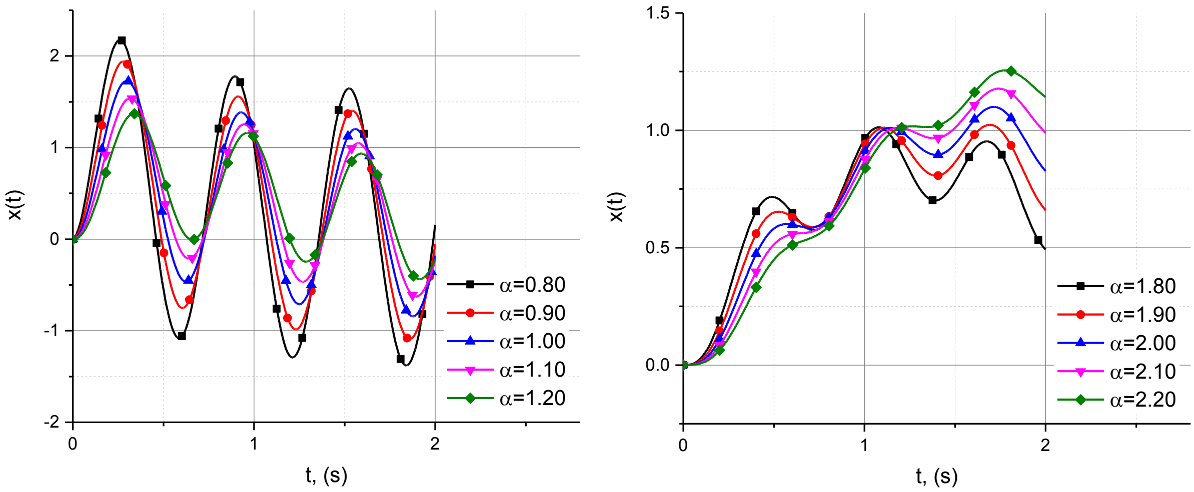

Fig. 2: Progression of vibrations of the system for $\alpha \approx 1$ *and* $\alpha \approx 2$.

The obtained results for $\alpha = 1$ and $\alpha = 2$ are compliant with the results of analytical calculations described by the following dependencies

$$x(t) = \frac{a\omega A0}{a^2\omega^2 + b^2} e^{-\frac{b}{a}\cdot t} + \frac{A0\left(b\sin(\omega\cdot t) - a\omega\cos(\omega\cdot t)\right)}{a^2\omega^2 + b^2} \tag{11}$$

$$x(t) = \frac{A0\omega}{\sqrt{\frac{b}{a}}\left(\omega^2 a - b\right)}\sin\left(\sqrt{\frac{b}{a}}\cdot t\right) - \frac{A0}{\left(\omega^2 a - b\right)}\sin(\omega\cdot t) \tag{12}$$

where (11) dependency presents the solution of the first model *(m = 0)*, and equation (12) is the solution of the second model *(c = 0)*.

The topic of ordinary differential equations is one of the most important subjects in mathematics and other sciences. However, there are no general methods to solve such equations. One of the most known methods to solve ordinary differential equations is the integral transform method. In this paper presented the Laplace transform were used to solve harmonic oscilator equations.

4. Conclusion

In the paper one replaced the two models of oscillating systems described by classical equations of harmonic oscillator, by one reduced model of fractional order. For the description of the above mentioned model one used the concept of derivative-integral of fractional order defined by Caputo. It is significant that both the derivative and integral are defined on the interval, in contrast to the classical definition of a derivative, which is defined at a point. Therefore, the applied mathematical model allows to determine the whole set of the courses of vibrations of the system, and not only, as it does in the classical theory of vibrations, for an equation of 1 or 2 order.

Reserences

Blasiak, M. (2016) Parametric analysis of piezoelectric transducer used for signal processing, in: Proc. Int. Conf. on Engineering Mechanics 2016 (ed. Zolotarev, I and Radolf, V.), Svratka, pp. 66-69.

Blasiak, S. (2016) Time-fractional heat transfer equations in modeling of the non-contacting face seals, International Journal of Heat and Mass Transfer, 100, , pp. 79-88. doi:10.1016/j.ijheatmasstransfer.2016.04.040

Blasiak, S., Takosoglu, J.E. and Laski, P.A. (2014) Heat transfer and thermal deformations in non-contacting face seals, Journal of Thermal Science and Technology, 9, 2, pp. JTST0011-JTST0011. doi:10.1299/jtst.2014jtst0011

Blasiak, S. and Zahorulko, A.V. (2016) A parametric and dynamic analysis of non-contacting gas face seals with modified surfaces, Tribology International, 94, , pp. 126-137. doi:10.1016/j.triboint.2015.08.014

Kaczorek, T. (2013) Application of the drazin inverse to the analysis of descriptor fractional discrete-time linear systems with regular pencils, International Journal of Applied Mathematics and Computer Science, 23, 1, pp. 29-33. doi:10.2478/amcs-2013-0003

Laski, P.A., Takosoglu, J.E. and Blasiak, S. (2014) A delta type closed kinematics chain with pneumatic muscle actuator manipulator, in: Proc. Int. Conf. on Engineering Mechanics 2014 (ed. Fuis, V.), Svratka, pp. 360-363.

Nowakowski, L., Miesikowska, M. and Blasiak, M. (2016) Speech intelligibility in the position of cnc machine operator, in: Proc. Int. Conf. on Engineering Mechanics 2016 (ed. Zolotarev, I and Radolf, V.), Svratka, pp. 422-425.

Stanislavsky, A.A. (2005) Twist of fractional oscillations, Physica A: Statistical Mechanics and Its Applications, 354, 1-4, pp. 101-110. doi:10.1016/j.physa.2005.02.033

Takosoglu, J.E., Laski, P.A., Blasiak, S., Bracha, G. and Pietrala, D. (2016) Determination of flow-rate characteristics and parameters of piezo valves, in: Proc. Int. Conf. Exp. Fluid Mech. 2016 (ed. Dancova, P.), Techn. Univ. Liberec, pp. 814-818.

Yonggang, K. and Xiu'e, Z. (2010) Some comparison of two fractional oscillators, Physica B: Condensed Matter, 405, 1, pp. 369-373. doi:10.1016/j.physb.2009.08.092

Zwierzchowski, J. (2016) Indsustrial robot vison system for moving spherical elements, in: Proc. Int. Conf. on Engineering Mechanics 2016 (ed. Zolotarev, I and Radolf, V.), Svratka, pp. 626-629.