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A SHORT WAVE LIMIT OF THE FREQUENCY EQUATION FOR PLANE-STRESS NONAXISYMMETRIC DISC MOTIONS

J. Červ*, F. Valeš, V. Adámek*****

Abstract: *It is proved that the general frequency equation for plane-stress nonaxisymmetric disc motions tends for the first mode of propagation and for wavelengths very short when compared with the disc radius to the secular equation for Rayleigh waves.*

Keywords: Frequency equation, Elastic disc, Rayleigh waves.

1. Introduction

The frequency equation for plane-stress nonaxisymmetric motions of an elastic disc (the disc boundary $r = r_1$ is assumed to be free of tractions) can be written as, see Cerv (1988),

$$\left\{\left[\frac{y^2}{2}-1+\frac{1}{\kappa}\right]J_\kappa(\kappa y\varphi)-\frac{y\varphi}{\kappa}J_{\kappa+1}(\kappa y\varphi)\right\}\cdot\left\{\left[\frac{y^2}{2}-1+\frac{1}{\kappa}\right]J_\kappa(\kappa y)-\frac{y}{\kappa}J_{\kappa+1}(\kappa y)\right\}-$$
$$\left\{\left[\frac{1}{\kappa}-1\right]J_\kappa(\kappa y)+yJ_{\kappa+1}(\kappa y)\right\}\cdot\left\{\left[\frac{1}{\kappa}-1\right]J_\kappa(\kappa y\varphi)+y\varphi J_{\kappa+1}(\kappa y\varphi)\right\}=0\,. \quad (1)$$

The symbols y and φ represent the dimensionless ratios

$$y=\frac{c}{c_2}\,,\;\; \varphi=\frac{c_2}{c_3}=\sqrt{\frac{1-\mu}{2}}\,, \quad (2)$$

where c_2 and c_3 are the velocities of shear and dilatational waves, respectively. Poisson's ratio is denoted by μ, c is the phase velocity. The parameter κ is the dimensionless wavenumber and it can be written as

$$\kappa=\frac{2\pi}{\lambda}r_1\,, \quad (3)$$

where r_1 is the disc radius, λ is the wavelength. The symbol J_κ denotes the Bessel function of the first kind, order κ. The frequency equation (1) has an infinite number of discrete roots $y = c/c_2$, each corresponding to a particular mode of propagation. In the paper Cerv (1988) it is shown by means of a numerical procedure that the first mode which belongs to the first dispersion curve represents Rayleigh–type waves.

The aim of the paper is to find a simpler form of the equation (1) which could approximate the first mode of propagation for wavelength very short when compared with the disc radius.

* Assoc. Prof. Jan Červ, PhD.: Institute of Thermomechanics AS CR, v.v.i.; Dolejškova 5; 182 00, Prague; CZ, cerv@it.cas.cz

** František Valeš, PhD.: Institute of Thermomechanics AS CR, v.v.i.; Veleslavínova 11; 301 14, Pilsen; CZ, vales@it.cas.cz

*** Vítězslav Adámek, PhD.: NTIS – New Technologies for the Information Society, University of West Bohemia; Univerzitní 8; 306 14, Pilsen; CZ, vadamek@kme.zcu.cz

2. Problem solution

Let κ be a sufficiently large number *($\kappa = 2\pi r_1 / \lambda >> 1$)*. For these short wavelengths (r_1 being arbitrary but fixed) and for the first mode of propagation it holds, see Cerv (1988),

$$0 < y < 1 \; . \tag{4}$$

From (2) it also follows

$$0 < \varphi < 1 \; . \tag{5}$$

The equation (1) may be rewritten into the form

$$\begin{aligned} &\left\{ \left[\frac{y^2}{2} - 1 + \frac{1}{\kappa} \right] - \frac{y\varphi}{\kappa} \frac{J_{\kappa+1}(\kappa y\varphi)}{J_\kappa(\kappa y\varphi)} \right\} \cdot \left\{ \left[\frac{y^2}{2} - 1 + \frac{1}{\kappa} \right] - \frac{y}{\kappa} \frac{J_{\kappa+1}(\kappa y)}{J_\kappa(\kappa y)} \right\} = \\ &= \left\{ \left[\frac{1}{\kappa} - 1 \right] + y \frac{J_{\kappa+1}(\kappa y)}{J_\kappa(\kappa y)} \right\} \cdot \left\{ \left[\frac{1}{\kappa} - 1 \right] + y\varphi \frac{J_{\kappa+1}(\kappa y\varphi)}{J_\kappa(\kappa y\varphi)} \right\} . \end{aligned} \tag{6}$$

It is evident that estimates of the ratios of the Bessel functions in (6) have to be determined.

2.1. Asymptotic representation of $J_{\kappa+1}(\kappa y) / J_\kappa(\kappa y)$, $J_{\kappa+1}(\kappa y\varphi) / J_\kappa(\kappa y\varphi)$

If α is any fixed and positive number and κ is large and positive, the following asymptotic expansion of $J_\kappa(\kappa \cdot \mathrm{sech}\alpha)$ is valid, see Watson (1966),

$$J_\kappa(\kappa \cdot \mathrm{sech}\alpha) \approx \frac{\exp\left[\kappa \left(\tanh\alpha - \alpha \right) \right]}{\sqrt{2\pi\kappa \cdot \tanh\alpha}} \sum_{m=0}^{\infty} \left\{ \frac{\Gamma\left(m + \frac{1}{2} \right)}{\Gamma\left(\frac{1}{2} \right)} \cdot \frac{A_m}{\left(\frac{1}{2} \kappa \cdot \tanh\alpha \right)^m} \right\} , \tag{7}$$

where $A_0 = 1$, $A_1 = \frac{1}{8} - \frac{5}{24} \coth^2 \alpha$,.... Taking only the first term of this expansion and writing

$$\mathrm{sech}\alpha = \frac{1}{\cosh\alpha} = y \, , \tag{8}$$

we have for $\kappa >> 1$

$$J_\kappa(\kappa y) \approx \frac{\exp\left[\kappa \left(\sqrt{(1 - y^2)} - \mathrm{arcsech}\, y \right) \right]}{\sqrt{(2\pi\kappa)} \cdot \left(\sqrt{(1 - y^2)} \right)^{1/2}} \; . \tag{9}$$

From (8) it is clear that for $\alpha > 0$ it holds $0 < y < 1$, i.e., the condition (4) is fulfilled. The corresponding formula for $J_{\kappa+1}([\kappa + 1] y)$ can be then derived from (9). We obtain

$$J_{\kappa+1}([\kappa + 1] y) \approx \frac{\exp\left[(\kappa + 1) \cdot \left(\sqrt{(1 - y^2)} - \mathrm{arcsech}\, y \right) \right]}{\sqrt{\left[2\pi (\kappa + 1) \right]} \cdot \left(\sqrt{(1 - y^2)} \right)^{1/2}} \; . \tag{10}$$

Let $\hat{y}$ be a new variable which is given by

$$\hat{y} = \frac{(\kappa + 1)}{\kappa} y \; . \tag{11}$$

For any fixed y, $0 < y < 1$, we can assign a number $\kappa_0 >> 1$ such that for every $\kappa > \kappa_0$ we have

$$0 < \hat{y} < 1 \; . \tag{12}$$

Substituting (11) into (10) leads to

$$J_{\kappa+1}(\kappa\hat{y}) \approx \frac{\exp\left[(\kappa+1)\cdot\left(\sqrt{(1-\left(\frac{\kappa}{\kappa+1}\right)^2\hat{y}^2)}-\operatorname{arcsech}\left(\frac{\kappa}{\kappa+1}\hat{y}\right)\right)\right]}{\sqrt{2\pi}\sqrt{(\kappa+1)}\cdot\left(\sqrt{(1-\left(\frac{\kappa}{\kappa+1}\right)^2\hat{y}^2)}\right)^{1/2}} . \tag{13}$$

Denoting denominator in (13) as D one gets after a small algebra $D=\sqrt{2\pi\kappa}\left(\sqrt{1+\frac{2}{\kappa}+\frac{1}{\kappa^2}-\hat{y}^2}\right)^{1/2}$.

Neglecting $2/\kappa$, $1/\kappa^2$ in D we receive for a sufficiently large κ the asymptotic expression for D

$$D \cong \sqrt{2\pi\kappa}\left(\sqrt{1-\hat{y}^2}\right)^{1/2} . \tag{14}$$

Let P be the exponent in numerator of (13). For P we get

$P=\kappa\left(\sqrt{1+\frac{2}{\kappa}+\frac{1}{\kappa^2}-\hat{y}^2}\right)-\kappa\cdot\operatorname{arcsech}\left(\frac{1}{1+1/\kappa}\hat{y}\right)-\operatorname{arcsech}\left(\frac{1}{1+1/\kappa}\hat{y}\right)$, and for a sufficiently large κ it may be written as

$$P \cong \kappa\left[\sqrt{1-\hat{y}^2}-\operatorname{arcsech}\hat{y}\right]-\operatorname{arcsech}\hat{y} . \tag{15}$$

In view of the expressions (14) and (15), the approximation (13) may be rewritten in the form

$J_{\kappa+1}(\kappa\hat{y}) \approx \frac{\exp\left[\kappa\left[\sqrt{1-\hat{y}^2}-\operatorname{arcsech}\hat{y}\right]\right]}{\sqrt{2\pi\kappa}\left(\sqrt{1-\hat{y}^2}\right)^{1/2}}\cdot\exp\left(-\operatorname{arcsech}\hat{y}\right)$, and by using the approximation (9), we obtain

$$J_{\kappa+1}(\kappa\hat{y}) \approx J_{\kappa}(\kappa\hat{y})\cdot\exp\left(-\operatorname{arcsech}\hat{y}\right) . \tag{16}$$

Taking the expression (8) and using the following identity for the inverse hyperbolic functions, as may be seen in the book of Rektorys (1968), we get for arbitrary z, $0<z\le 1$

$$\operatorname{arcsech} z = \operatorname{arccosh}\frac{1}{z} = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) . \tag{17}$$

The substitution of (17) into (16) yields (for $z=\hat{y}$) $J_{\kappa+1}(\kappa\hat{y}) \approx J_{\kappa}(\kappa\hat{y})\cdot\left(\frac{1}{\hat{y}}+\left[\frac{1}{\hat{y}^2}-1\right]^{1/2}\right)^{-1}$ and after a simple algebra we have

$$\frac{J_{\kappa+1}(\kappa\hat{y})}{J_{\kappa}(\kappa\hat{y})} \approx \frac{1-\sqrt{1-\hat{y}^2}}{\hat{y}} . \tag{18}$$

The approximation (18) is true for a sufficiently large κ and for any fixed $\hat{y}$, $0<\hat{y}<1$. It is evident that terms having the argument $\kappa y\varphi$ (see (6)) can be treated in the same manner. Therefore, one gets the similar approximation

$$\frac{J_{\kappa+1}(\kappa\hat{y}\varphi)}{J_{\kappa}(\kappa\hat{y}\varphi)} \approx \frac{1-\sqrt{1-\left(\hat{y}\varphi\right)^2}}{\hat{y}\varphi} . \tag{19}$$

Now we may return to the equation (6). If we substitute (18) and (19) (with original variable y) into the equation (6), and then neglect small quantities as $\kappa \to +\infty$, we obtain

$$\left[\frac{y^2}{2}-1\right]^2 - \sqrt{1-y^2}\cdot\sqrt{1-\varphi^2 y^2} = 0 \ . \tag{20}$$

The equation (20) may be considered to be an approximation of the general equation (1) (or (6)) for the first mode of propagation as $\kappa \to +\infty$. In view of (2), the equation (20) may have the form

$$\left[\left(\frac{c}{c_2}\right)^2 - 2\right]^2 - 4\left[1-\left(\frac{c}{c_2}\right)^2\right]^{1/2}\cdot\left[1-\left(\frac{c}{c_3}\right)^2\right]^{1/2} = 0 \ . \tag{21}$$

The equation (21) is the well-known secular equation for Rayleigh waves in isotropic elastic 2D continuum, see Graff (1975). Secular equations for Rayleigh waves in anisotropic media are studied in the paper by Cerv & Plesek (2013).

3. Conclusions

It is proved that the general frequency equation (1) for plane-stress nonaxisymmetric disc motions tends to the secular equation for Rayleigh waves for the first mode of propagation and for wavelength very short when compared with the disc radius r_1. The former results reached in Cerv (1988) by a numerical procedure were corroborated by this study.

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