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CALCULATION OF SHELL STRUCTURES BY USING LAYERED MODEL

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Abstract: *The DKT plate element and the plane element with rotational degrees of freedom are employed to define a shell element. Moreover, layered model is taken into account to include possible nonlinear behaviour across a cross-section of a shell. The paper explains principles of layered model and interaction between the elements used. The model and the shell element are implemented into the SIFEL solver using finite element method.*

Keywords: Shell element, DKT plate element, Plane element with rotational degrees of freedom, Layered model, Finite element method.

1. Introduction

When the topic of shell structures is concerned, nonlinear behaviour of a material is often desirable to be taken into consideration. Although this subject can be resolved by using 3D elements together with a nonlinear material model, with respect to the planar character of shell structures, it is preferred to use 2D modelling tools to find a solution to the problem. In this regard, a possible way to involve nonlinearity in a calculation is by adding layered model to the process (Hu and Schnobrich, 1991).

The authors have continued to explore the topic of layered model within the SIFEL environment (Krejčí, Koudelka and Kruis, 2011) and are now extending the work from previous years focused on the application of layered model solely on plate structures to the application on shell elements.

For purpose of this application, a triangle shell element was created by combining two separate elements – the DKT plate element and the triangle plane element with rotational degrees of freedom. The idea behind the shell element was to use two, in SIFEL already developed, separate elements without any changes and to form an interaction between them so they could simulate the shell behaviour. Although the idea itself seems very simple, several subsidiary issues has emerged, especially with regard to integration points.

The first part of the paper is dedicated to basic description of layered model. Principles of the DKT and the rotational plane element are then explained followed by the examination of the elements' interaction. At the end, several issues that have been dealt with are mentioned in detail.

2. Layered model

When describing deformation of a shell structure, the following vectors can be adopted

$$\boldsymbol{\varepsilon_0} = \left\{\varepsilon_{0,x},\ \varepsilon_{0,y},\ \gamma_{0,xy}\right\}^T, \quad \boldsymbol{\kappa} = \left\{\kappa_x,\ \kappa_y,\ \kappa_{xy}\right\}^T, \tag{1}$$

where $\boldsymbol{\varepsilon_0}$ is the vector of middle plane strain and $\boldsymbol{\kappa}$ represents the vector of curvatures. If the statement, that straight lines normal to the middle plane remain straight and normal to the middle plane after deformation, is accepted, the development of deformation $\boldsymbol{\varepsilon} = \left\{\varepsilon_x, \varepsilon_y, \gamma_{xy}\right\}^T$ across the *z*-coordinate can be expressed as follows

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$$\boldsymbol{\varepsilon}(z) = \boldsymbol{\varepsilon}_0 + z\boldsymbol{\kappa}\,. \tag{2}$$

When layered model is employed, a shell structure is decomposed into small layers (Fig. 1) and Eq. 2 can be rewritten into the following form

$$\boldsymbol{\varepsilon}_j = \boldsymbol{\varepsilon}_0 + z_j\boldsymbol{\kappa} \tag{3}$$

where $\boldsymbol{\varepsilon_j}$ represents strain components of the j-th layer and z_j is the distance of the j-th layer from the middle plane. Each layer is considered to be in the plane stress state and by using the appropriate stiffness matrix $\boldsymbol{Dj}$, stress components of the j-th layer $\boldsymbol{\sigma_j}$ can be obtained

$$\boldsymbol{\sigma}_j = \boldsymbol{D}_j\,\boldsymbol{\varepsilon}_j \tag{4}$$

The integration of the stress components over the layer thickness yields resultant forces

$$\boldsymbol{n} = \{n_x, n_y, n_{xy}\}^T,\ \ \boldsymbol{m} = \{m_x, m_y, m_{xy}\}^T \tag{5}$$

that are obtained as the summation of contributions from all layers. The single contribution from the j-th layer to the stress resultant forces is formulated as follows

$$\boldsymbol{n}_j = t_j\,\boldsymbol{\sigma}_j,\ \ \boldsymbol{m}_j = z_j t_j\,\boldsymbol{\sigma}_j \tag{6}$$

And by combining all contributions and substituting Eq. 3 and Eq. 4 into Eq. 5, the final stress resultant forces are attained

$$\begin{Bmatrix}\boldsymbol{n}\\ \boldsymbol{m}\end{Bmatrix} = \sum_j \begin{bmatrix} t_j\boldsymbol{D}_j & z_j t_j\boldsymbol{D}_j \\ z_j t_j\boldsymbol{D}_j & z_j^2 t_j\boldsymbol{D}_j\end{bmatrix}\begin{Bmatrix}\boldsymbol{\varepsilon}_0\\ \boldsymbol{\kappa}\end{Bmatrix} \tag{7}$$

It is important to notice here that layered model enables to use various stiffness matrices for layers. For even distribution of stiffness across a cross-section, off-diagonal members of the matrix in Eq. 7 are equal to zero. Only when stiffness is distributed unequally, off-diagonal members are take into account.

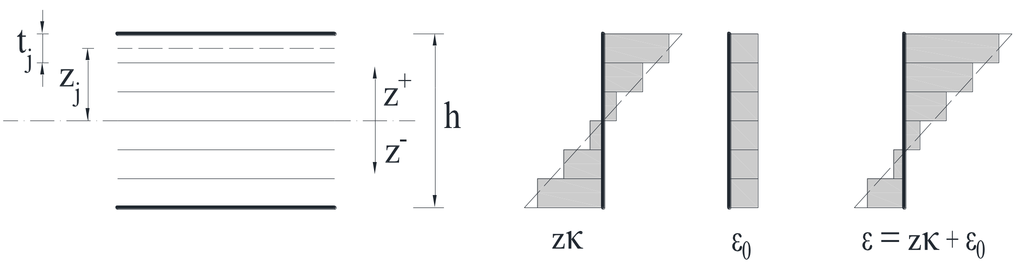

Fig. 1: Decomposition of a structure into layers.

3. Elements used and their interaction

A general shell element holds 6 degrees of freedom at each node – 3 displacements *u*, *v*, *w* and 3 rotations $\varphi_x, \varphi_y, \varphi_z$. In the case of layered model, it is assumed that the triangle shell element is assembled from the DKT element and the plane element with rotational degrees of freedom (Jirásek and Bažant, 2002), (Bittnar and Šejnoha, 1996).

In this interaction, the DKT element is intended to represent plate behaviour of a shell and the distribution of deflection w and rotations φ_x, φ_y is from the element obtained. The plane element is employed to calculate the remaining displacements *u*, *v* and the rotation φ_z. In addition, the curvatures $\kappa_x, \kappa_y, \kappa_{xy}$ are computed within the DKT element and the strains $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ are received from the plane element, which are then used, with regard to Eq. 1, as the middle plane strains.

The class of the shell element (primary element) is in SIFEL created in the form of a coordinator between the plate element and the plane element (secondary elements). The idea is to take already developed plate and plane elements and use them in the algorithm as individual units without any changes. The work done by the shell element can be summarized into the following points:

- assembles the stiffness matrix
- transforms values from global to local character and vice versa
- collects data from the core calculation and redistributes them into the secondary elements and vice versa
- gathers calculation data from the secondary elements and generate the final output

Although these are fairly obvious procedures, the authors would like to introduce some difficulties that have come out through the implementation of the shell element.

4. Stiffness matrix

The stiffness matrix of an element can be expressed in finite element method as

$$\boldsymbol{K} = \int_V \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \, dV \tag{8}$$

where $\boldsymbol{B}$ represents the matrix of shape functions derivations and $\boldsymbol{D}$ is the matrix of material stiffness. Let us focus only on the expression after the integration mark in Eq. 8. While considering a shell element assembled by one plate element and one plane element, it can be stated that

$$\boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_N^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}_M^T \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_N & \boldsymbol{D}_{NM} \\ \boldsymbol{D}_{MN} & \boldsymbol{D}_M \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_N & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}_M \end{bmatrix} \tag{9}$$

where $\boldsymbol{B}_N$ is the matrix of shape functions derivations for a plane element and $\boldsymbol{B}_M$ for a plate element. $\boldsymbol{D}_N$ then represents the material stiffness matrix of a plane element and $\boldsymbol{D}_M$ of a plate element. The off-diagonal members $\boldsymbol{D}_{NM}$ and $\boldsymbol{D}_{MN}$ can be interpreted as a relation between the two elements. If linear elasticity is considered, the off-diagonal members are neglected and Eq. 9 can be, after multiplication, rewritten into the following form

$$\begin{bmatrix} \boldsymbol{B}_N^T \boldsymbol{D}_N \boldsymbol{B}_N & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}_M^T \boldsymbol{D}_M \boldsymbol{B}_M \end{bmatrix} \tag{10}$$

which is exactly as a plate and a plane element are simply put together. However, when some form of nonlinearity across a cross-section is employed, the off-diagonal members cannot be ignored and Eq. 9 is possible to modify into

$$\begin{bmatrix} \boldsymbol{B}_N^T \boldsymbol{D}_N \boldsymbol{B}_N & \boldsymbol{B}_N^T \boldsymbol{D}_{NM} \boldsymbol{B}_M \\ \boldsymbol{B}_M^T \boldsymbol{D}_{MN} \boldsymbol{B}_N & \boldsymbol{B}_M^T \boldsymbol{D}_M \boldsymbol{B}_M \end{bmatrix}. \tag{11}$$

To fit this idea on layered model, the material stiffness matrices $\boldsymbol{D}_N$, $\boldsymbol{D}_M$, $\boldsymbol{D}_{NM}$ and $\boldsymbol{D}_{MN}$ can be interpreted in regards of Eq. 7 as follows

$$\boldsymbol{D}_N = \sum t_j \boldsymbol{D}_j; \quad \boldsymbol{D}_M = \sum z_j^2 t_j \boldsymbol{D}_j; \quad \boldsymbol{D}_{NM} = \boldsymbol{D}_{MN} = \sum z_j t_j \boldsymbol{D}_j \, . \tag{12}$$

After the numerical integration, it is possible to achieve the stiffness matrix of the shell element that is created by a plate and a plane element and equipped with layered model.

5. Integration points and nonlinear calculation

When performing a calculation on integration points, it is necessary to be aware of the fact that the chosen plane element includes more integration points than the DKT plate element. Although both elements use the 3-point integration system, the plane element performs the integration twice therefore possesses two sets of the same 3 integration points. Given that layered model determines stress resultant forces on every integration point and at the same time uses both middle plane strains and curvatures, the so-called artificial integration points are established to deal with the issue. These integration points collect both strain vectors and perform calculation of stress resultant forces. The results are then distributed to the appropriate integration points on particular elements. The artificial integration points hold only the composition of an integration point but they are not used for the integration of nodal forces involved in the equilibrium equations. They just perform the calculation of stress resultant forces and further generated data inside are eventually used for creating the output. It should be also noted that the artificial points are created in respect of the 3-point integration system. Instead of performing calculation of stress resultant forces 3 times (once on the DKT element and twice on the plane element), it is done only once and results are spread over the remaining integration points.

As mentioned at the beginning, adopting layered model can be a suitable way to include nonlinearity. When a shell structure is imaginarily divided into layers, each layer is assigned its own strain components according to Eq. 3. While the plane stress state is considered, any material model can be attached to every layer, thus creating space for involving nonlinearity and at the same time keeping the problem in 2D. As stated above, all calculations are done on the artificial integration points where all nonlinear parameters for layers are stored and printed out to the output in case of a successful calculation.

As an example of a nonlinear calculation using layered model, the results from a concrete rectangular slab considering plastic yielding in layers are shown in Fig. 2. The slab is supported simply alongside the upper and the side edges while the lower edge is fixed. The slab is loaded uniformly and the values of plastic multiplier γ, that indicate the amount of plastic yielding at the area, can be observed from Fig. 2.

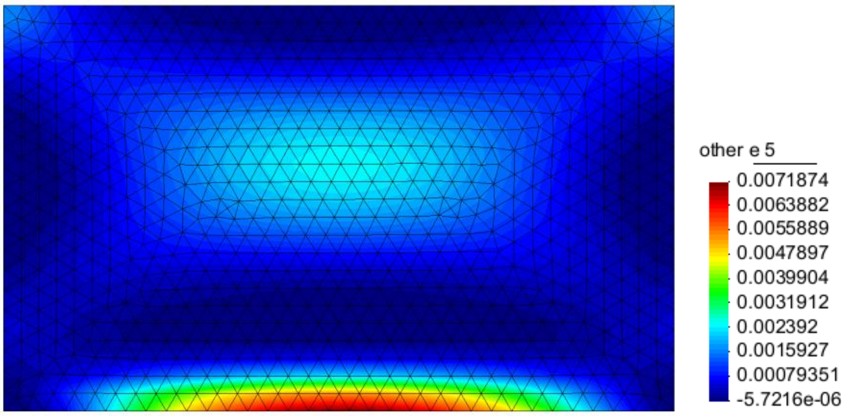

Fig. 2: Development of plastic multiplier γ in a rectangular slab [-].

6. Conclusions

The application of layered model to the shell element assembled from the DKT plate element and the plane element with rotational degrees of freedom has been presented in the paper. Difficulties and principles regarding the implementation of layered model and the shell element itself has been mentioned.

Acknowledgement

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