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LYAPUNOV EXPONENTS – PRACTICAL COMPUTATION

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Abstract: *The Lyapunov exponents serve as numerical characteristics of dynamical systems, which measure possible divergence of nearby trajectories of the solution. In this way they express dependence of the dynamical system on initial conditions. However, the value of Lyapunov exponents consists in their ability to characterise deterministic chaos. The limiting process intrinsic in the definition of Lyapunov exponents, unfortunately, seriously complicates their computation. The short paper presents an overview of difficulties in numerical approaches to enumeration of Lyapunov exponents or at least the largest one and shows a promising method based on QR decomposition of the system Jacobian.*

Keywords: Lyapunov exponent, Dynamical system, Non-linear system.

1. Introduction

The collocation "deterministic chaos" seems to be a contradiction in itself. However, in the theory of dynamical systems it refers to systems whose results are reproducible but (almost) unpredictable. No mathematical definition of the term deterministic chaos is universally accepted yet and chaos is usually characterised by its properties. The reasonably simple working definition due to Strogatz (2000) states: Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions. The Lyapunov exponents (LE) provide a numerical measure of the last condition of the Strogatz's definition: sensitivity on initial conditions. Thus, a strictly positive LE in dissipative systems can be, under additional conditions, regarded as an indicator of deterministic chaos (Politi 2013, Dieci 2011). For other examples of their applications see, e.g., (Ott 1993). It is worth noting that in most applications it is sufficient to approximate only a subset of LE, e.g., the largest one.

Consider the m dimensional non-linear differential system

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x}) , \quad \mathrm{x}(0) = \mathrm{x}_0 . \tag{1}$$

Let the right-hand side f is a smooth function and solution x_{x_0} of (1) exists for given initial condition x_0 and $t \geq 0$. Using two terms of Taylor expansion of $f(x)$ can be easily derived that the infinitesimal perturbations δ to a trajectory x_{x_0} are described by the linearized equation

$$\dot{\delta} = \mathbf{J}f\left(x_{x_0}\right)\delta , \tag{2}$$

where $\mathbf{J}f(x)$ be the Jacobian of the right-hand side f in (1). Eq. (2) is linear ordinary differential equation and its stability can be examined using traditional means. In dynamical systems, evolution of size of the perturbation δ is governed by the relation

$$\|\delta(t)\| = \mathrm{e}^{\lambda_1 t}\|\delta(0)\| , \tag{3}$$

where λ_1 is the largest LE. The relation (3) is usually supposed to serve as a formula for calculation of λ_1:

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \ln\|\delta(t)\| , \tag{4}$$

Regarding the whole spectrum of LE, the exact definition is more complicated. Dieci et al. (2011) define upper and lower LE as follows. Let $A(t) = \mathbf{J}f\left(x_{x_0}\right)$ and define numbers $\mu_i, \nu_i, i = 1, \ldots, m$ such that

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$$\mu_i = \limsup_{t\to\infty} \frac{1}{t}\ln\|Y(t)e_i\|\,, \quad \nu_i = \limsup_{t\to\infty} \frac{1}{t}\ln\|Z(t)e_i\|, \tag{5}$$

where $m \times m$ matrices $Y(t), Z(t)$ are the solutions of

$$\dot{Y}(t) = A(t)Y(t), Y(0) = Y_0, \quad \dot{Z}(t) = -A(t)^{\mathrm{T}}Z(t), Z(0) = Z_0\,, \quad Y_0, Z_0 \in \mathbb{R}^{m\times m} \text{ regular} \tag{6}$$

and e_i is the i-th standard unit vector. Such μ_i, ν_i, which minimize $\sum_{i=1}^{m}\mu_i\,, \sum_{i=1}^{m}\nu_i$ for all possible initial conditions Y_0, Z_0 are called *upper* or *lower Lyapunov exponent* and denoted λ_i^u, λ_i^l, respectively. If $\lambda_i^u, = \lambda_i^l = \lambda_i$ for all i the system is called regular. Politi (2013) uses slightly different yet equivalent definition. Vast majority of publications assumes regularity of the underlying system. This assumption, e.g., enables validity of the basic computational rule (4). Also the present work will follow this practice.

2. Numerical methods

Numerical approximation of LE is often introduced using discrete maps instead of continuous dynamical system. The discretized continuous system can be usually regarded as a discrete map; however, such an approach imposes restrictions on discretization parameters and, moreover, changes properties of the original dynamical system.

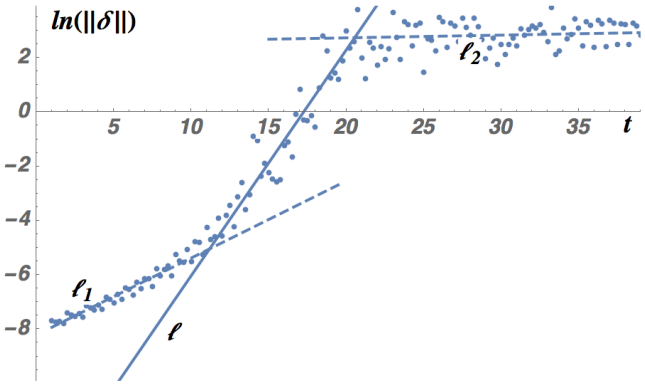

Fig. 1: Numerical approximation of largest LE of the Lorenz attractor.

Most of the introductory texts start and end with estimation of the largest Lyapunov exponent $\lambda_1$using the limiting approach similar to Eq. (4). Danger of this simple approach is illustrated in Fig. 1, inspired by Strogatz (2000). It shows typical evolution of separation of two adjacent trajectories, $\|\delta_0\| = 10^{-3}$ for the Lorenz system $\sigma = 10, b = 8/3, r = 28$. The correct value $\lambda_1 \approx 0.9$ is well approximated by the slope the line ℓ. Fig. 1 points out two weaknesses of the approach (4). For $t < 10$ is the mean slope of $\|\delta\|$ shown as ℓ_1 and for $t > 20$ as ℓ_2. The line ℓ_1 reflects influence of initial conditions, for different initial conditions can be either almost horizontal or disappear at all. It can be eliminated in the real computation by selecting the initial condition on the attractor. The plateau ℓ_2, on the other hand, appears when separation of the trajectories is comparable to the size of the attractor and cannot be eliminated. Fig. 1 shows that algorithmization of the method based on (4) could be a cumbersome task.

Besides the initial condition, other factors influencing estimation of the largest LE can be: discretization parameter of the differential solver, size of initial separation of trajectories $\|\delta_0\|$ and number of averaged trials for random initial perturbations of size $\|\delta_0\|$. The discretization parameter or choice of (stable!) integration method does not seem to influence results too much, the procedure works well even with adaptive methods provided the samples of individual trajectories match. On the other hand, size of initial perturbation influences results significantly. From Tab. 1 can be seen that too large δ_0 results in an underestimated value of λ_1. The second row of the table shows values determined from a single random perturbation and rows 3 and 4 list average values and variances computed from 100 different initial perturbations of size $\|\delta_0\|$, respectively.

Tab. 1: Influence of size of initial perturbation on the estimation of the largest LE.

$\|\delta_0\|$	**0.1**	**0.01**	**0.001**	**0.0001**	**0.00001**	**10^{-6}**
single λ_1	0.48004	0.68527	0.72202	0.96167	0.93266	0.95062
mean λ_1	0.27817	0.56540	0.81150	0.93027	0.94803	0.94875
variance λ_1	0.02018	0.01525	0.00700	0.00050	0.00053	0.00004

The same approach can be practically used when several or all LE are to be determined. The procedure has to be augmented by regular reorthogonalization, in the similar manner as in the well-known subspace iteration method for computing eigenvalues. A large variety of methods is based this approach, mainly based on theoretical and practical papers by Galgani et al. (1980) and Benettin et al. (1980). Example of a

good practical implementation is available due to Sandri (1996). When using this approach, the computation can easily exploit parallel architecture of current computers (Tange 2011).

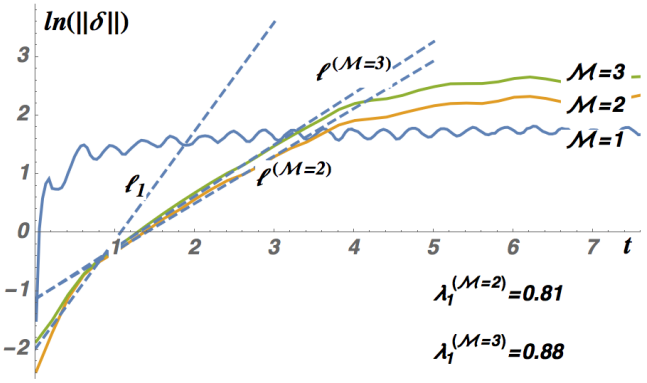

Fig. 2: Approximation of largest LE of the Lorenz attractor using discrete data for three values of the embedded dimension $\mathcal{M} = 1, 2, 3$.

The approach based on Eq. (4) can be also used in the case when there are available only the experimental or other discrete data. The simple implementation accompanying paper (Wolf 1985) or the monograph (Kantz and Schreiber 2004) provide basic functionality only and cannot distinguish chaotic and stochastic data. The more advanced procedure described by Rosenstein et al. (1993) is similarly based on identifying of different yet similar sections in the experimental data, which are used subsequently to simulate separating of close trajectories. Although the algorithm is relatively simple, its application depends on several more or less heuristic parameters. These are namely embedded dimension $\mathcal{M}$, reconstruction delay $\mathcal{J}$ and number of data available $\mathcal{N}$. While the parameter $\mathcal{N}$ is usually fixed, determination of optimal values for $\mathcal{M}$ and $\mathcal{J}$ can be tricky, for discussion and hints see the cited paper.

The example results for the x-coordinate of the Lorenz system from the previous page for three different values of $\mathcal{M}$ are shown in Fig. 2. Data were obtained using a fixed step fourth order RK integrator with $\Delta t = 0.01$ and delay was set to $\mathcal{J} = 18 \mathrel{\widehat{=}} 0.18$s. The solid curves show time dependency of the averaged values of $\ln\|\delta\|$. As in Fig. 1, the curves for $\mathcal{M} = 2{,}3$ exhibit linear growth for ca. $1 < t < 3.5$, giving a good estimate of λ_1. In accordance with Rosenstein's results the case $\mathcal{M} = 1$ gave no usefull result. However, presence of false linear slope ℓ_1 for both $\mathcal{M} = 2, 3$ lowers credibility of obtained results.

The recent and advanced approaches to stability assessment are based on the idea of maintaining the orthogonal transformation of the perturbation coordinates during integration, e.g., Náprstek (2014). Lyapunov (1992) showed in his thesis that for regular systems the LE may be extracted as the limit of the time average of the diagonal elements of the upper triangular coefficient matrix. If the coefficient matrix (denoted $\boldsymbol{B}$) in the (regular) system (2) is triangular, then its LE can be determined as

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \int_0^t B_{ii}(s)\, \mathrm{d}s \,. \qquad (7)$$

The main idea of the work presented by Dieci et al. (2011) is based on transformation of the general Jacobian $\mathbf{J}f(x_{x_0})$ to an upper triangular matrix $\boldsymbol{B}$ using a time-dependent orthogonal QR transformation. The algorithms developed by Dieci et al. offer computation of several or all LE of a (not necessarily regular) system. The authors claim that their packages LESLIS, LESLIL, LESNLS and LESNLL are mature enough to provide a scientific community with a mean for reliable, yet cautious, estimation of LE. All methods implemented in the package start from a full rank initial condition $Y_0 \in \mathbb{R}^{n\times m}$, where $n \le m$ is a number of LE to be computed. It is advised to select the Y_0 at random, this is needed to guarantee that all possible growth behaviour is represented in the columns of $Y(t)$ and that the n most dominant exponents will emerge. In each step $t_i \to t_{i+1}$, the methods perform (i) integration, (ii) orthogonalization, (iii) update of LE estimates. This division of the general procedure is simple and allows a free hand in implementation of individual tasks. The package offers several special integration schemes, including those which preserve orthogonality. Unlike the traditional methods, this package uses adaptive steplength based on local error estimate for both increased efficiency and accuracy. The time-variable orthogonalization is implemented as either discrete or continuous QR factorization. The both methods are equivalent in case of exact arithmetic, however, their performance will depend on a particular problem. As an alternative solution strategy, the user is enabled to use his own particular integrator (e.g., a stiff one), approximate each step the solution trajectory using polynomials and let the linear version of the code to evaluate the LE estimates. This latter approach could be also used for estimation of LE of a discrete (experimental) data series.

To illustrate the method, the non-linear version of the code was employed to compute the LE of the same Lorenz system as above. Namely, the procedure LESNLS was used, which require explicit formulation of

the Jacobian of the system. The computed LE were: $\lambda_1 = 0.9095$, $\lambda_2 = -0.000696$, $\lambda_3 = -14.5755$ and this values correspond well with other sources. The computed Lyapunov exponents as a function of time in a semilog scale are shown in Fig. 3.

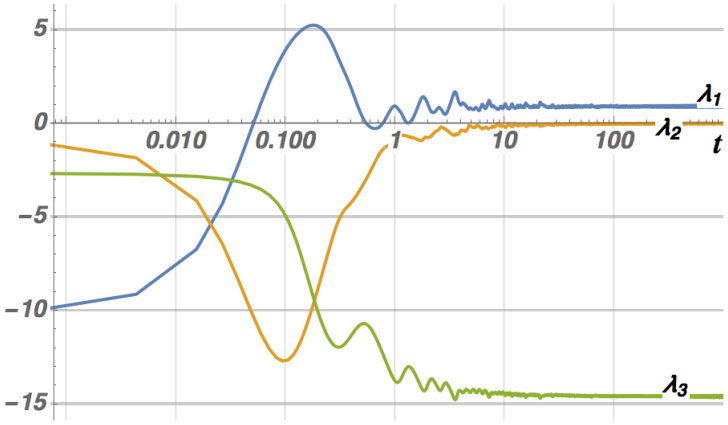

Fig. 3:Convergence of all three LE of the Lorenz attractor using the QR procedure LESNLS.

3. Conclusions

The Lyapunov exponents play an irreplaceable role in characterization of dynamical systems behaviour. By the very nature of the limiting process intrinsic in the definition of Lyapunov exponents, their approximation is bound to be limited in extent, and perhaps the approximations themselves may be considered of dubious validity. This fact was shown in the first part of the text, where several definition-based approaches were discussed. Although there exist good implementations of such methods, stress was put upon illustrative examples of difficulties present in the nature of estimation of Lyapunov exponents for continuous systems and experimental data. On the other hand, it seems that the approach based on the orthogonal QR factorization of the Jacobian can serve as a reasonable and robust method.

Acknowledgement

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