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# EXACT 3-NODED TIMOSHENKO BEAM FINITE ELEMENT WITH ENHANCED STRAIN FIELD – A MAGIC ROLE OF GAUSS POINTS

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**Abstract:** *The present paper is dedicated to the idea of enhanced strain field in development of multi-node Timoshenko beam finite element. It is proved that it is possible to obtain exact stiffness matrix for 3-noded beam element with carefully selected substitute strain field and enrichment points. The selection can be recommended for more sophisticated analysis of plates and shells.*

**Keywords: Timoshenko beam, Finite element, Enhanced strain, Gauss points.**

## 1. Introduction

Beams, plates and shells are widely applied in engineering structures. However, there are several difficulties with discretization procedures employed to calculate such elements – among others, locking phenomena and spurious strains (Gilewski, 2005), which are of major importance. Development of high-performance finite elements for moderately thick beams, plates and shells dates back to 1970. It was followed by hundreds of publications concerning different conceptions for finite elements that were supposed to deal with the appearing difficulties.

According to the present state of the art, one of the best conception is a formulation of the finite element with enhanced strain field (Bathe et al., 1989; Brezzi et al., 1989; Dvorkin and Bathe, 1984; Simo and Rifai, 1990). In this formulation, spurious strains are approximated independently of displacements, and after that they are “sewed” with strains resulting from displacements in the selected amount of points. The conception has now reliable mathematical foundations in the form of theorems of solution existence and uniqueness, or several variational formulations (Bathe et al., 1989; Borja, 2008; Brezzi et al., 1989; Militello and Felippa, 1990). The idea of enhanced strain fields is still developed for shell elements (Cesar, 2002; Choóścielewski and Witkowski, 2006).

The present study deals with construction of three-noded finite elements with enhanced strain fields as a base study for plate/shell multi-node elements. Arbitrary enrichment points are applied. It has been proved that using Gauss-Legendre points, an exact three-noded finite element is obtained.

## 2. Algorithm

Taking into consideration a Timoshenko beam problem, the following values may be defined: displacements $\mathbf{u}(\xi)=\{w(\xi),\phi(\xi)\}$, strains $\boldsymbol{\varepsilon}(\xi)=\{\kappa(\xi),\gamma(\xi)\}$ and stresses (internal forces) $\boldsymbol{\sigma}(\xi)=\{M(\xi),Q(\xi)\}$. In the above notation, a dimensionless coordinate system is used: $\xi = x/a$, where $a$ is a characteristic dimension (equal to the half of the element length in further considerations). The geometric relations between strains and displacements are as follows:

$$\kappa = -\frac{1}{a}\frac{d\phi}{d\xi} = \mathbf{D}_1\mathbf{u};\ \ \mathbf{D}_1 = \begin{bmatrix} 0 & -\frac{1}{a}\frac{d}{d\xi} \end{bmatrix}, \tag{1}$$

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$$\gamma = \frac{1}{a}\frac{dw}{d\xi} - \phi = \mathbf{D}_2\mathbf{u};\ \ \mathbf{D}_2 = \begin{bmatrix} \frac{1}{a}\frac{d}{d\xi} & -1 \end{bmatrix}, \tag{2}$$

and the constitutive equations are: $M = EJ\kappa$, $Q = kGA\gamma$, where $M$ is a bending moment, $Q$ is a shear force, $E$ is Young's modulus, $J$ is a moment of inertia, $k$ is a shear correction factor, $G$ is a shear modulus, $A$ is a cross-sectional area.

A standard approximation of an element displacement field $\mathbf{u}(\xi) = \mathbf{N}(\xi)\mathbf{q}$ and a strain field:

$$\kappa(\xi) = \mathbf{B}_1(\xi)\mathbf{q},\ \ \mathbf{B}_1 = \mathbf{D}_1\mathbf{N},\ \ \gamma(\xi) = \mathbf{B}_2(\xi)\mathbf{q},\ \ \mathbf{B}_2 = \mathbf{D}_2\mathbf{N} \tag{3}$$

are used. Transverse shear strains (3), as a function of dimensionless coordinates $\alpha_i$, can be calculate at enrichment points: $x_i = \alpha_i a$ for $i = 1, 2,..., N$:

$$\gamma(\alpha_i) = \mathbf{B}_2(\alpha_i)\mathbf{q}\,. \tag{4}$$

An independent interpolation of the enhanced transverse shear strain field is applied:

$$\gamma_{enhanced} = C_1 + C_2\xi + C_3\xi^2 + ... + C_N\xi^{N-1} = \mathbf{p}^T(\xi)\mathbf{C}\,, \tag{5}$$

where $\mathbf{p}(\xi) = \{1, \xi, \xi^2,..., \xi^{N-1}\}$, $\mathbf{C} = \{C_1, C_2, C_3,..., C_N\}$.

Assuming equality of strain fields at each enrichment point:

$$\mathbf{p}^T(\alpha_i)\mathbf{C} = \mathbf{B}_2(\alpha_i)\mathbf{q};\ i = 1, 2,..., N, \tag{6}$$

for all of the following enrichment points we obtain:

$$\mathbf{PC} = \tilde{\mathbf{B}}\mathbf{q}\,, \tag{7}$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}^T(\alpha_1) \\ \mathbf{p}^T(\alpha_2) \\ \cdots \\ \mathbf{p}^T(\alpha_N) \end{bmatrix},\ \ \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_2(\alpha_1) \\ \mathbf{B}_2(\alpha_2) \\ \cdots \\ \mathbf{B}_2(\alpha_N) \end{bmatrix}. \tag{8}$$

Now the interpolation constants $\mathbf{C}$ may be calculated as:

$$\mathbf{C} = \mathbf{P}^{-1}\tilde{\mathbf{B}}\mathbf{q}\,. \tag{9}$$

The enhanced transverse shear strain field can be presented in the following form:

$$\gamma_{enhanced}(\xi) = \mathbf{p}^T(\xi)\mathbf{P}^{-1}\tilde{\mathbf{B}}\mathbf{q} = \mathbf{B}_2^{enhanced}(\xi)\mathbf{q}\,, \tag{10}$$

where

$$\mathbf{B}_2^{enhanced}(\xi) = \mathbf{p}^T(\xi)\mathbf{P}^{-1}\tilde{\mathbf{B}}\,. \tag{11}$$

The total strain field is constructed as follows:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \kappa \\ \gamma_{enhanced} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1(\xi) \\ \mathbf{B}_2^{enhanced}(\xi) \end{bmatrix}\mathbf{q} = \mathbf{B}_{enhanced}\mathbf{q}\,. \tag{12}$$

As a next step, one can construct an element stiffness matrix according to the standard procedure employed in a displacement formulation of FEM, with the integration over the element length:

$$\mathbf{K} = \int_\Omega \mathbf{B}_{enhanced}^T\mathbf{E}\mathbf{B}_{enhanced}\,d\Omega\,. \tag{13}$$

## 3. Three-noded exact beam finite element

A three-noded beam finite element, *2a* in length, with natural degrees of freedom is considered:

$$\mathbf{q} = \{w_1 \quad \phi_1 \quad w_2 \quad \phi_2 \quad w_3 \quad \phi_3\}. \tag{14}$$

Applying polynomial interpolation, the shape function matrix can be define as follows:

$$\mathbf{N}(\xi)=\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}, \tag{15}$$

where: $N_1(\xi)=\frac{1}{2}\xi(\xi-1),\ N_2=1-\xi^2,\ N_3(\xi)=\frac{1}{2}\xi(\xi+1).$

For two symmetrically located enrichment points $\alpha_1=-\alpha,\ \alpha_2=\alpha$ and linear enhanced strain field $\gamma_{enhanced}=C_1+C_2\xi$, the following element stiffness matrix is obtained:

$$\mathbf{K}=\begin{bmatrix}
\frac{7H}{6a} & -H\frac{3\alpha^2+2}{6} & -\frac{4H}{3a} & H(\alpha^2-1) & \frac{H}{6a} & -H\frac{3\alpha^2-2}{6} \\
-H\frac{3\alpha^2+2}{6} & \frac{7EJ+Ha^2(3\alpha^4+1)}{6a} & \frac{2H}{3} & \frac{-4EJ-3Ha^2\alpha^2(\alpha^2-1)}{3a} & H\frac{3\alpha^2-2}{6} & \frac{EJ+Ha^2(3\alpha^4-1)}{6a} \\
-\frac{4H}{3a} & \frac{2H}{3} & \frac{8H}{3a} & 0 & -\frac{4H}{3a} & -\frac{2H}{3} \\
H(\alpha^2-1) & \frac{-4EJ-3Ha^2\alpha^2(\alpha^2-1)}{3a} & 0 & \frac{8EJ+6Ha^2(\alpha^2-1)^2}{3a} & -H(\alpha^2-1) & \frac{-4EJ-3Ha^2\alpha^2(\alpha^2-1)}{3a} \\
\frac{H}{6a} & H\frac{3\alpha^2-2}{6} & -\frac{4H}{3a} & -H(\alpha^2-1) & \frac{7H}{6a} & H\frac{3\alpha^2+2}{6} \\
-H\frac{3\alpha^2-2}{6} & \frac{EJ+Ha^2(3\alpha^4-1)}{6a} & -\frac{2H}{3} & \frac{-4EJ-3Ha^2\alpha^2(\alpha^2-1)}{3a} & H\frac{3\alpha^2+2}{6} & \frac{7EJ+Ha^2(3\alpha^4+1)}{6a}
\end{bmatrix} \tag{16}$$

The crucial aspect is to select the location of enrichment points. A criterion applied in this study consists in comparing maximum deflection of a cantilever beam loaded with a concentrated force (Fig. 1a), obtained using one finite element with the enhanced strain field, and an exact solution from the Timoshenko beam theory.

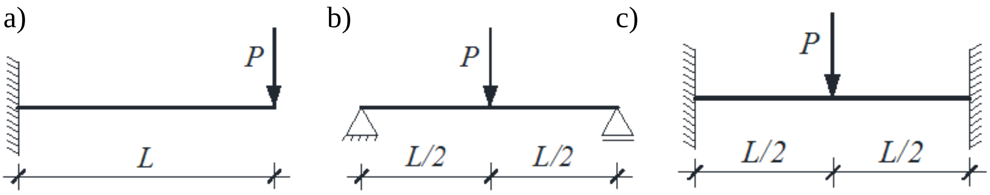


*Fig. 1: Timoshenko beam: a) Cantilever; b) Simply supported; c) Clamped.*

The results are as follows:

- exact solution:

$$w_{\max}^{exact}=\frac{2Pa^3(4-3\mu)}{3EJ(1-\mu)},\quad \mu=\frac{3\gamma}{1+3\gamma},\quad \gamma=\frac{EJ}{Ha^2}, \tag{17}$$

- enhanced solution – 1 finite element:

$$w_{\max}^{enhanced}=\frac{2Pa}{H}+\frac{Pa^3(7-6\alpha^2+3\alpha^4)}{2EJ}. \tag{18}$$

Comparison of the received solutions, (17) and (18), leads to biquadratic equation:

$$3\alpha^4-6\alpha^2+\frac{5}{3}=0. \tag{19}$$

Solutions of the Eq. (19) are superconvergence points of transverse shear strains enrichment:

$$\alpha^{1,2}=\pm\sqrt{\frac{1}{3}},\quad \alpha^{3,4}=\pm\sqrt{\frac{5}{3}}. \tag{20}$$

The first two solutions correspond to the location of Gauss-Legendre points. The third and the fourth solution are located outside the finite element. In all cases, we obtain the exact solution of a cantilever beam problem, using one finite element with the enhanced strain field. Mesh refining does not influence on solution.

The exact stiffness matrix is following:

$$\mathbf{K}_{e2} = \begin{bmatrix} \frac{7H}{6a} & -H\frac{1}{2} & -\frac{4H}{3a} & -H\frac{2}{3} & \frac{H}{6a} & H\frac{1}{6} \\ -H\frac{1}{2} & \frac{21EJ+4Ha^2}{18a} & \frac{2H}{3} & \frac{-108EJ-4Ha^2}{81a} & -H\frac{1}{6} & \frac{3EJ-2Ha^2}{18a} \\ -\frac{4H}{3a} & \frac{2H}{3} & \frac{8H}{3a} & 0 & -\frac{4H}{3a} & -\frac{2H}{3} \\ -H\frac{2}{3} & \frac{-108EJ-4Ha^2}{81a} & 0 & \frac{72EJ+24Ha^2}{27a} & H\frac{2}{3} & \frac{-12EJ-6Ha^2}{9a} \\ \frac{H}{6a} & -H\frac{1}{6} & -\frac{4H}{3a} & H\frac{2}{3} & \frac{7H}{6a} & H\frac{1}{2} \\ H\frac{1}{6} & \frac{3EJ-2Ha^2}{18a} & -\frac{2H}{3} & \frac{-12EJ-6Ha^2}{9a} & H\frac{1}{2} & \frac{21EJ+4Ha^2}{18a} \end{bmatrix}. \quad (21)$$

It is possible to check that the calibration of enrichment points on the cantilever beam (matrix (21)) gives also exact results for other boundary conditions. One can receive exact solution for simply supported (Fig. 1b) as well as clamped (Fig. 1c) beams with the use of single finite element.

## 4. Conclusions

The present paper presents construction of stiffness matrix for a 3-noded finite element of a moderately thick beam with enhanced strain fields. Applying linear substitute strains and enrichment points: $\alpha^{1,2} = \pm\sqrt{1/3}$ or $\alpha^{3,4} = \pm\sqrt{5/3}$, the exact three-noded finite element is received. The enrichment points can be recommended for more sophisticated analysis of plates and shells.

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