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CALDERÓN'S INVERSE PROBLEM IN CIVIL ENGINEERING

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Abstract: *In specific fields of research such as treatment of historical structures, medical imaging, material science, geophysics and others, it is of particular interest to perform only a non-intrusive boundary measurement. The idea is to obtain a comprehensive information about the material properties inside the domain under consideration while maintaining the test sample intact. This contribution is focused on such problems i.e. synthesizing a physical model of interest with a boundary inverse techniques. The forward model is represented by a basic time dependent diffusion equation with Finite Element (FE) discretization and the parameters are subsequently recovered using a modified Calderon problem principle, numerically solved by a regularized Gauss-Newton method. We provide a basic framework, implementation details and modification of general constrains originally derived for a standard setup of Calderon problem. The proposed model setup was numerically verified for various domains, load conditions and material field distributions. Both steady-state and time dependent cases are studied.*

Keywords: Boundary inverse, Finite Element Method, Calderón's problem, Heat transfer.

1. Introduction

In this contribution we propose two linear models describing a heat transfer and an inverse method based on Neuman-to-Dirichlet (NTD) operator. Although the idea of boundary inverse method using electric current dates back to 1930s in geophysics it has gained more attention must later in 1980s as a medical imaging technique, i.e. Electrical Impedance Tomography (EIT). The first rigorous formulation of this problem is the most commonly attributed to Argentinian mathematician Alberto Calderón who formed his thoughts in his foundational paper (Calderón 1980). Further development and proofs of uniqueness of the solution were given in (Somersalo, et al., 1992 and Brown, 1997). The basic procedure in EIT is following: by stimulating electrodes attached on a body surface with different injection patterns and simultaneously measuring the resulting potentials on these electrodes, one can with the knowledge of domain shape, electrode impedance and applied current uniquely recover the isotropic conductivity field (Calderón 1980).

2. Forward models

The presented models will play a fundamental role since each will be repeatedly evaluated in the inverse process and will also substitute an experiment, i.e. will be used to generate artificial measurements. The steady-state model represents a straight forward, single-parameter model that closely relates to the classical concept of Calderón problem, while the time-dependent model is essentially a two-parameter based and has a wider spread of use in real world applications.

2.1. Steady-state heat transfer

Following the same principles mentioned in (Calderón, 1980), one can obtain a similar set of equations for a steady-state heat transfer, allowing a more general boundary conditions. The governing equations may therefore take a following form

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$$
\begin{cases}
\nabla \cdot (\lambda(x)\nabla u(x)) = 0, & x \in \Omega \\
\alpha^{(i)}\left(u_0^{(i)}(x) - u(x)\right) = f_T, & x \in \partial\Omega_T^{(i)} \backslash e_l \\
u(x) + r_l \lambda \frac{\partial u}{\partial n}(x) = T_l, & x \in e_l \\
\lambda \frac{\partial u}{\partial n}(x) = f_N, & x \in \partial\Omega_N^{(i)} \backslash e_l
\end{cases} \tag{1}
$$

where u is a temperature, λ is a thermal conductivity, L is the number of electrodes, r_l is an electrode resistance coefficient, e_l is l-th electrode and T_l is the l-th stimulation pattern. The environment factors are α being the heat transfer coefficient, u_0 is the outside temperature and f_N are prescribed fluxes with corresponding boundary subsets $\partial\Omega_T$ and $\partial\Omega_N$ respectively.

Contrary to electrostatics, the consequences of aforementioned conditions are following relaxed assumptions under which the solution can be proved (Somersalo, et al., 1992 and Cheng, et al., 1989):

Assumption 1. *The conductivity λ, contact resistances r_l and transfer coefficients $\alpha^{(i)}$ satisfy following*

$$
\begin{gathered}
(i)\ \lambda \in L^\infty(\Omega; R), \inf_{x\in\Omega} \lambda(x) = \lambda_- > 0, \\
(ii)\ 0 < r_l^- \le r_l \le r_l^+ < \infty, l = 1, \dots, L, \\
(iii)\ 0 < \alpha_-^{(i)} \le \alpha^{(i)} \le \alpha_+^{(i)} < \infty, \forall i.
\end{gathered}
$$

In this settings, we assume the accessible boundary Γ_m is completely captured with a thermal camera or with an array or discrete thermometers. Each active electrode is then consecutively, one after each other, heated to temperature T_l resulting into L thermal images of the observed boundary Γ_m.

2.2. Time dependent heat transfer

In real conditions it is, however, not an easy task to maintain a stable and steady state conditions. Not only the surrounding temperature will fluctuate, but for standard building materials like bricks, wood, etc. the steady state, after changing the loading conditions, is reached after several hours or days depending on the volumetric capacity, heat conductivity, material thickness and temperature change. Therefore, we intend to apply the identical principles used in a Calderón problem for time dependent models.

To capture a time dependent heat transfer, one can adapt following set of equations

$$
\begin{cases}
\rho c_p \frac{\partial u}{\partial t} - \nabla \cdot (\lambda \nabla u) = f, & x \in \Omega \\
\alpha^{(i)}\left(u_0^{(i)}(x) - u(x)\right) = f_T, & x \in \partial\Omega_T^{(i)} \\
u = f_D, & x \in \partial\Omega_D^{(i)} \\
\lambda \frac{\partial u}{\partial n}(x) = f_N, & x \in \partial\Omega_N^{(i)}
\end{cases} \tag{2}
$$

where ρ is volumetric mass density, c_p is specific heat capacity and $\partial\Omega_{N,T,D}^{(i)}$ are non-intersecting subsets of boundary $\partial\Omega^{(i)}$ in i-th loading condition with corresponding environmental factors u_0, $\alpha^{(i)}$, $f_{T,D,N}$. In order to maintain NTD sensing, the set of equations in is subjected to following constrain

Assumption 2. *Let Γ_m be a subset of boundary $\partial\Omega$ that is being observed. Then*

$$
\Psi = \left(\partial\Omega_N^{(i)} \cup \partial\Omega_T^{(i)}\right) : (\Psi \cap \Gamma) \notin \emptyset, \forall i,
$$

must hold, i.e. the boundary subjected to measurements must contain at least some Neumann conditions.

Conditions as indicated in assumption 1 must be also met, i.e. *(ii,iii)* have to be extended for ρ and c_p.

In this settings, the model is intended to only rely on the environment natural factors without stimulating electrodes. Also the data from thermal cameras are recorded continuously on the accessible boundary.

3. Numerical solution of the inverse problem

Results in section 4 share the following regularized Gauss-Newton iteration scheme (Holder, 2004)

$$
\sigma_{k+1} = \sigma_k + \delta\sigma_k \tag{3}
$$

$$\delta\sigma_k = \left(\boldsymbol{J}_k^{\mathrm{T}}\boldsymbol{J}_k + \alpha_k \boldsymbol{L}^{\mathrm{T}}\boldsymbol{L}\right)^{-1}\left(\left(\boldsymbol{J}_k^{\mathrm{T}}\left(\boldsymbol{u}_r - F(\sigma_k)\right)\right) - \alpha_k \boldsymbol{L}^{\mathrm{T}}\boldsymbol{L}(\sigma_k - \sigma_r)\right) \quad (4)$$

where $f(\sigma) \in R^{vw}$ represents a discrete NTD operator of a forward model with v being the number of measurement points and w is the number of experiments. The a priori measured quantity is stored in vector $\boldsymbol{u}_r \in R^{vw}$ and the regularization operator L is a pre-calculated Laplacian. In all cases the reference field $\sigma_r = \sigma_0 = 1$. From our experience, the most reliable choice of hyper-parameter α_k was the one used in Levenberg-Marquardt regularization in a following form

$$\alpha_k = \max(max(\boldsymbol{J}_k^T\boldsymbol{J}_k)) \quad (5)$$

Jacobian $\boldsymbol{J}_k$ was updated in each iteration and was calculated numerically in a following way

$$J_i^{(jkl)} = \frac{\partial u_{jk}}{\partial \sigma_i^l} \quad (6)$$

where $\boldsymbol{J}_i$ is a third-order tensor in *i*-th iteration, indexes *jk* are representing measurement nodes in FE mesh and individual measurements respectively. Index *l* identifies a conductivity change on *l*-th element.

4. Results

In this section we investigate the models under various conditions. Specifically, we consider partial data reconstruction and different material properties.

From Fig. 2 and Fig. 3 one can notice that the rightmost figures suffer from an insufficiency of data leading into major artefacts in reconstructed material fields. Also the non-smooth material field was more difficult to recover, which was most evident in the leftmost Fig. 2 and Fig. 3.

Results for a single parameter steady-state simulations were generated in approximately 10 Gauss-Newton iterations, whereas the two-parameter time-dependent problem took 72 iterations. Despite the inaccuracies in certain situations, e.g. insufficiency of data, non-smooth material field, the Gauss-Newton method proved to be stable and reliable solver for such tasks.

Steady-state heat transfer model

The foregoing results were generated for following boundary conditions: $f_T = 10 \cdot (u_i - u)$, where $u_1 = 30\,°\mathrm{C}$, $\mathrm{u}_2 = 15\,°\mathrm{C}$. The other parameters were chosen in a following way: $T_l = 10\,°\mathrm{C}$, $r_l = 0.01 \frac{°\mathrm{C\,m^2}}{\mathrm{W}}$.

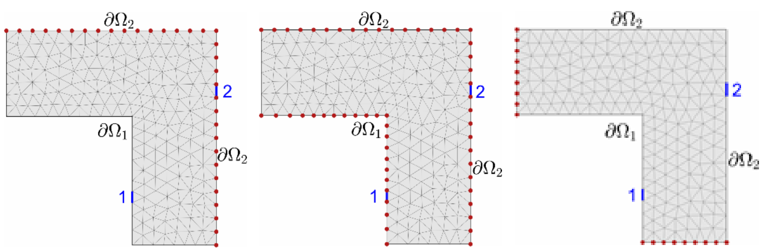

Fig. 1: A set of domains with different boundaries subjected to measurements. Red dots: measurement nodes, blue lines: electrodes.

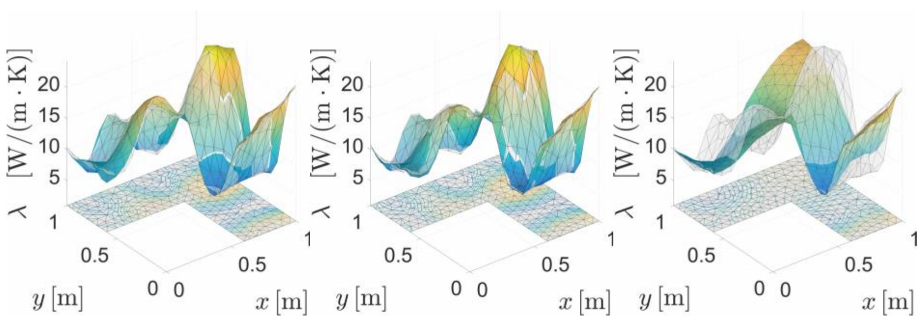

Fig. 2: Reconstruction: a smooth distribution. In grey: original field, in color: reconstructed field.

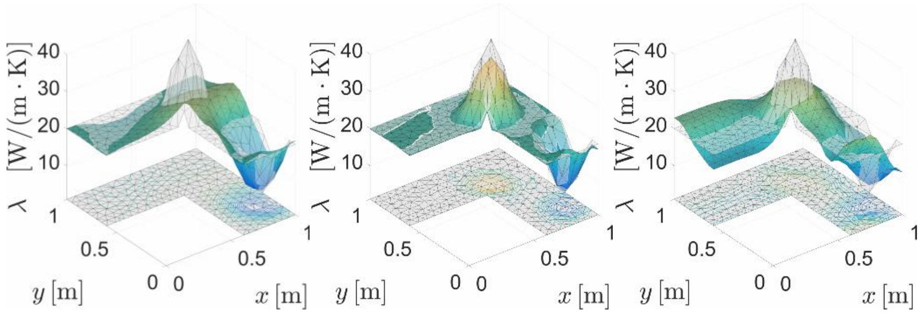

Fig. 3: Reconstruction: non-smooth distribution. In grey: original field, in color: reconstructed field.

Time-dependent heat transfer model

Assumed boundary conditions are captured in Fig. 4 with a following meaning: $f_T|_{\partial\Omega_i} = 10 \cdot (f_i - u)$

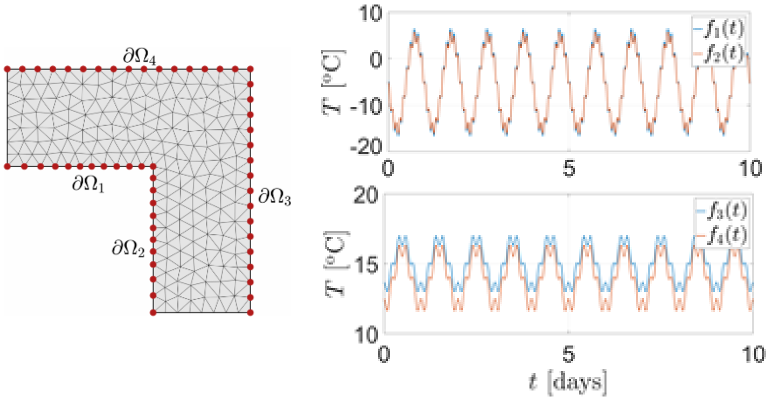

Fig. 4: Domain with oundary conditions.

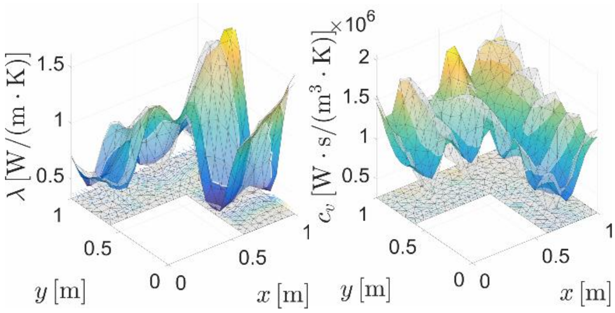

Fig. 5: Reconstruction: left: conductivity field, right: volumetric capacity.

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