23rd International Conference
ENGINEERING MECHANICS 2017
Svratka, Czech Republic, 15 – 18 May 2017

AXISYMMETRIC DEFORMATION OF AN ELASTIC MEDIUM WEAKENED BY AN ANNULAR CRACK

B. Kebli[*], Z. Baka[**]

Abstract: *The present work deals with the study of an axisymmetric flat annular crack problem in a semi-infinite elastic medium. The external surface is supposed without mechanical loads while the crack surface is subjected to a uniform internal pressure. The mixed boundary value problem is solved by the Hankel integral transforms method using the Boussinnesq's stress functions. Then the three-part mixed boundary value problem is converted to a system of coupled triple integral equations of Bessel functions. Furthermore, by applying some integral relations and involving the addition Gengenbauer formula, the system is reduced to a solution of an infinite algebraic system equations for determining the unknown coefficients development. The explicit formula for the stress intensity factors near the crack fronts are derived, by means of those coefficients.*

Keywords: Axisymmetric deformation, Mixed boundary value problem, Annular crack, Triple integral equations, Stress intensity factors.

1. Introduction

The stress analysis of crack problems is of fundamental interest for the study of initiation and propagation of fracture and failure in brittle material. The annular crack form is one of the defect can be met. To examine such a problem diverse analytical and approximate methods were proposed by the researcher. The earlier studies were presented by Moss et al. (1971), who develop an iterative approximate solution for the stress intensity factors near the crack fronts. The analysis presented by Shibuya et al. (1975), (1976), and Koizumi et al. (1977) employ an analytical method to reduce the governing triple integral equations to a solution of an infinite algebraic system. Mastrojannis et al. (1981) present an approximate solution of the axisymmetric problem of an annular crack embedded in an infinite elastic solid. By using the Betti's reciprocal theorem, Choi et al. (1982) derive the integral equations for a problem involving flat toroidal crack subjected to axial or torsional load. Selvadurai et al. (1985) reduce the annular three-part mixed boundary value problem to the solution of a system Fredholm integral equations which they solve by the parameter method.

The purpose of this paper is to examine an annular crack in a semi-infinite elastic medium, opened up by an internal pressure. The used method is inspired from the works of Shibuya et al. (1976) and Koizumi et al. (1977), which deal with a crack problem in an elastic solid. The analysis transforms the three-part mixed boundary value problem to a resolution of system the two triples integral equations, which is reduced directly to a solution of an infinite algebraic system equations, for determining the unknown coefficients development. The stress intensity factors are expressed in term of the series involving those coefficients.

2. Problem formulation

The annular crack is located on the $z = 0$ plane, with the inner and outer radii a and b, respectively, as shown in Fig. 1. The external surface of the infinite half-space medium $z = -h$ is supposed without mechanical load, while the crack is assumed under a uniform internal pressure p_0. Since the crack

[*] Prof. Belkacem Kebli, PhD.: Department of Engineering Mechanics, National Polytechnic School, El-Harrach; 16200, Alger; Algeria,belkacem.kebli@g.enp.edu.dz

[**] Eng. Zakaria Baka, PhD. Student:Department of Engineering Mechanics, National Polytechnic School, El-Harrach; 16200, Alger; Algeria, zakaria.baka@g.enp.edu.dz

generates a discontinuity in the medium, it is required to be divided into two regions: (1) denotes the upper one ($z \geq 0$), whereas (2) corresponds the lower region ($-h \leq z \leq 0$).

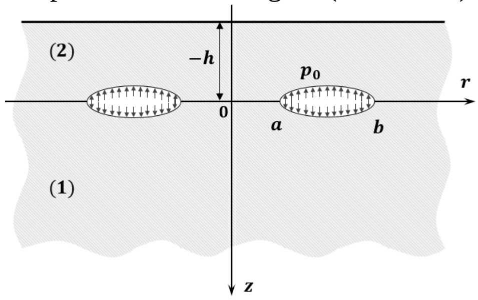

Fig. 1: Geometry of the problem.

The displacement and stress components should be reduced to zero as $r^2+z^2 \to \infty$ for $z \geq 0$. Furthermore, the appropriate boundary and continuity conditions can be formulated as follows

$$\sigma_z^{(2)}(r,-h) = \tau_{rz}^{(2)}(r,-h) = 0\,,\, r \geq 0 \tag{1}$$

$$\sigma_z^{(1)}(r,0^+) = \sigma_z^{(2)}(r,0^-) = -p_0\,, \quad \tau_{rz}^{(1)}(r,0^+) = \tau_{rz}^{(2)}(r,0^-) = 0\,, \quad a < r < b \tag{2}$$

$$\sigma_z^{(1)}(r,0^+) = \sigma_z^{(2)}(r,0^-)\,, \quad \tau_{rz}^{(1)}(r,0^+) = \tau_{rz}^{(2)}(r,0^-)\,, \quad 0 \leq r < a \text{ or } r > b \tag{3}$$

$$u_r^{(1)}(r,0^+) = u_r^{(2)}(r,0^-)\,, \quad u_z^{(1)}(r,0^+) = u_z^{(2)}(r,0^-)\,, \quad 0 \leq r \leq a \text{ or } r \geq b \tag{4}$$

where u_r, u_z are the radial and axial displacement components, σ_z and τ_{rz} are the components of normal and shear stresses along z and r, respectively.

3. Derivation of the integrals equations

For an axisymmetric problem with cylindrical coordinate system of r and z, in the absence of body forces, the general solution of the Lamé equilibrium equations for the two non-vanishing displacement components $u_r^{(i)}(r, z)$ and $u_z^{(i)}(r, z)$, $I = 1, 2$ can be expressed in terms of Boussinesq's stress functions $\varphi_0^{(i)}(r, z)$ and $\varphi_3^{(i)}(r, z)$, cf. Shibuya, 1975. They satisfy the following harmonic equations

$$\Delta\varphi_0^{(i)} = \Delta\varphi_3^{(i)} = 0 \tag{5}$$

where Δ is Laplace's operator stated to the cylindrical coordinate system. Moreover, the stresses components can be clearly represented by tatter stress functions on the basis of Hooke's law. The solution of equations (5) can be obtained by applying the Hankel transform method of the zeroth order and its inverse. Thus, it can be expressed in a general form as

$$\varphi_0^{(1)}(r,z) = \int_0^\infty A e^{-\lambda z} J_0(\lambda r)\, d\lambda\,, \quad \varphi_3^{(1)}(r,z) = \int_0^\infty B e^{-\lambda z} J_0(\lambda r)\, d\lambda \tag{6}$$

$$\varphi_0^{(2)}(r,z) = \int_0^\infty \left(C e^{-\lambda z} + D e^{\lambda z}\right) J_0(\lambda r)\, d\lambda\,, \quad \varphi_3^{(2)}(r,z) = \int_0^\infty \left(E e^{-\lambda z} + F e^{\lambda z}\right) J_0(\lambda r)\, d\lambda \tag{7}$$

where J_0 denotes the zeroth order Bessel function of the first kind. The auxiliary functions A, B, C, D, E and F with respect to λ are to be determined from the appropriate boundary and continuity conditions. Then we can get the displacement and stress components in term of those arbitrary functions. By means of the boundary and the continuity conditions (1), (2) and (3), a system of four equations is obtained for calculating the unknown functions A, B, C, D, E and F. From this system, it is expedient to reduce the six unknown functions into only two. Meanwhile, using the resulting formulas and inserting them into the conditions (4) and (2), leads to a system of coupled triples integral equations, which depend on the both Bessel functions of the first kind of zeroth and first order. This system may be stated in closed form as

$$\begin{aligned} &\int_0^\infty \left[E e^{2\lambda h} - (2\lambda h - 1) F\right] J_0(\lambda r)\, d\lambda = 0, \quad r \leq a, r \geq b \\ &\int_0^\infty \lambda\left[\left(e^{2\lambda h} - 2\lambda h - 1\right) E - \left(e^{-2\lambda h} + 2\lambda h - 1\right) F\right] J_0(\lambda r)\, d\lambda = 2p_0, \quad a < r < b \end{aligned} \tag{8}$$

$$\int_0^\infty \left[Ee^{2\lambda h} - (2\lambda h + 1)F\right] J_1(\lambda r)\,d\lambda = 0, \quad r \le a, r \ge b$$
$$\int_0^\infty \lambda\left[\left(e^{2\lambda h} + 2\lambda h - 1\right)E + \left(e^{-2\lambda h} - 2\lambda h - 1\right)F\right] J_1(\lambda r)\,d\lambda = 0, \quad a < r < b \tag{9}$$

Following Shibuya et al. (1976) and Koizumi et al. (1977) works, for solving the above system we use the integrals

$$\int_0^\infty G_n(\lambda) J_0(\lambda r)\,d\lambda \quad \text{and} \quad \int_0^\infty Z_n(\lambda) J_1(\lambda r)\,d\lambda \tag{10}$$

The first integral equations of (8) and (9) are then satisfied by taking

$$Ee^{2\lambda h} - (2\lambda h - 1)F = \sum_{n=1}^{\infty} a_n G_n(\lambda) \text{ and } Ee^{2\lambda h} - (2\lambda h + 1)F = \sum_{n=1}^{\infty} b_n Z_n(\lambda) \tag{11}$$

where a_n and b_n are unknown coefficients. Next, from the equations (11), by expressing E and F in term of the two unknown coefficients, and substituting in the second equations of (8) and (9), and using the addition Gengenbauer formula, after some algebra, one gets an algebraic system stated in a closed matrix form as

$$\sum_{n=1}^{\infty}\int_0^\infty \left(A_{mn}a_n + B_{mn}b_n\right) = 2p_0\delta_{1,m}$$
$$\sum_{n=1}^{\infty}\int_0^\infty \left(C_{mn}a_n + D_{mn}b_n\right) = 0 \qquad m=1, 2, ... \tag{12}$$

where $\delta_{1,m}$ denotes the Kronecker delta, and

$$A_{mn} = \int_0^\infty \left[-\left(2\lambda^2h^2 + 2\lambda h + 1\right)e^{-2\lambda h} + 1\right] G_m(\lambda) G_n(\lambda)\,d\lambda$$
$$B_{mn} = \int_0^\infty 2\lambda^2h^2e^{-2\lambda h} G_m(\lambda) Z_n(\lambda)\,d\lambda$$
$$C_{mn} = \int_0^\infty 2\lambda^2h^2e^{-2\lambda h} G_n(\lambda) Z_m(\lambda)\,d\lambda \tag{13}$$
$$D_{mn} = \int_0^\infty \left[-\left(2\lambda^2h^2 - 2\lambda h + 1\right)e^{-2\lambda h} + 1\right] Z_m(\lambda) Z_n(\lambda)\,d\lambda$$

Consequently, the governing integral equations were reduced to the solution of algebraic system equations (12). On the basis of these results, the displacements and stresses components as well as the stress intensity factors given by the following formulas

$$\begin{bmatrix} K_{Ia} \\ K_{IIa} \end{bmatrix} = \lim_{r \to a^-} \sqrt{a-r} \begin{bmatrix} \sigma_z(r,0) \\ \tau_{rz}(r,0) \end{bmatrix}$$
$$\begin{bmatrix} K_{Ib} \\ K_{IIb} \end{bmatrix} = \lim_{r \to b^+} \sqrt{r-b} \begin{bmatrix} \sigma_z(r,0) \\ \tau_{rz}(r,0) \end{bmatrix} \tag{14}$$

can be derived.

4. Results and discussion

The unknown coefficients a_n and b_n discussed in previous section are determined by solving the system (12), whereas the infinite integrals in (13) can be evaluated numericaly. Here, it had been tested that fifteen terms of coefficients are sufficient to get numericaly good results of stress intensity factors.

Fig. 2 shows the variations of normalized stress intensity factors corresponding of mode I and II near the crack fronts in function of *a/b* for different values of *h/b*. It is noted that the stress intensity factors are giving their large values when $a/b \to 0$ with a thinner thickness $h/b \to 0$, decrease with increasing one of the preceding parameters. Additionally, it is clear that the SIFs corresponding of mode I is always greater than the SIFs of mode II. The SIFs corresponding of mode I in the inner crack is greater than the other, however, the opposite behavior occurs for the SIFs corresponding of mode II. Moreover, with increasing the layer thickness $h/b \to \infty$, the obtained results are in good agreement with those reported in the current literatures, such as the works of Shibuya et al. (1975), Koizumi et al. (1975), Mastrojannis et al. (1981), Choi et al. (1982) and Selvadurai et al. (1983).

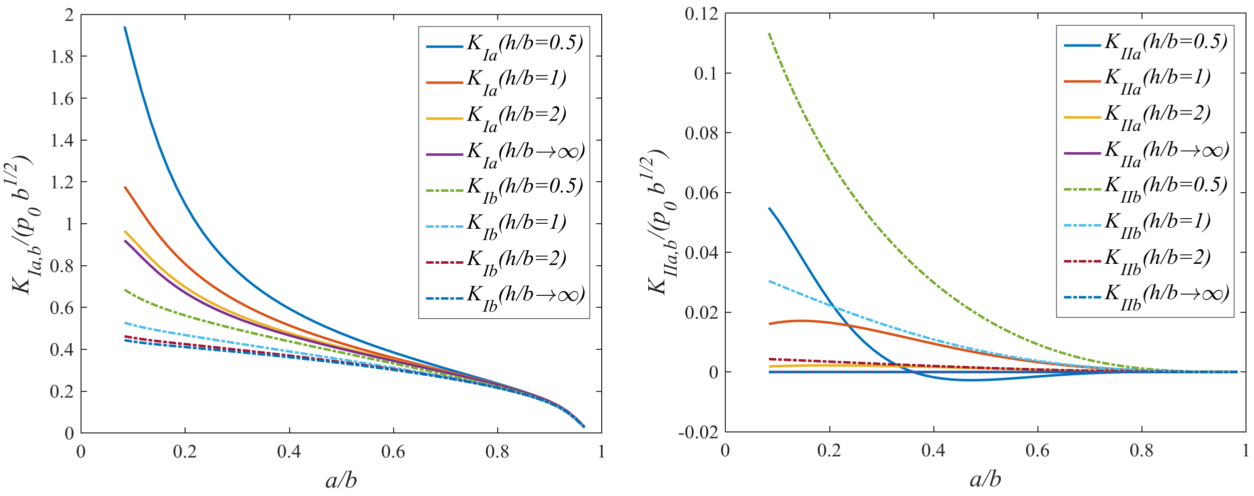

Fig. 2: Variations of stress intensity factors corresponding of mode I and II at the inner and the outer radii of the crack against a/b for various values of h/b.

5. Conclusions

This work studies the axisymmetric problem related to the internal loading of an annular crack located in a semi-infinite elastic medium. The mixed boundary value problem was reduced to the solution of an infinite algebraic system equations. A closed form solution of stress intensity factors were obtained. As a result, we can analyze their variations, to figure out the depth effect of the thickness and the radii on the crack propagation.

Acknowledgement

This work was supported by the Laboratoire de Génie Mécanique et développement (LGMD), ENP.

References

Choi, I. and Shield, R.T. (1982) A note on a flat toroidal crack in an elastic isotropic body. International Journal of Solids and Structures, 18, 6, pp. 479-486.

Koizumi, T., Shibuya, T. and Nakahara, I. (1977) A crack problem on a slab with an annular crack subjected to pressure. Bulletin of JSME, 20, 139, pp. 17-23.

Mastrojannis, E.N. and Kermanidis, T.B. (1981) An Approximate solution of the annular crack problem. International Journal for Numerical Methods in Engineering, 17, pp. 1605-1611.

Moss, L.W. and Kobayashi, A.S. (1971) Approximate analysis of axisymmetric problems in fracture mechanics with application to a flat toroidal crack. International Journal of Fracture Mechanics, 7, 1, pp. 89-99.

Selvadurai, A.P.S. and Singh, B.M. (1985) The annular crack problem for an isotropic elastic solid. Quarterly Journal of Mechanics and Applied Mathematics, 38, 2, pp. 233-243.

Shibuya, T., Koizumi, T. and Nakahara, I. (1976) The axisymmetric stress distribution in an infinite elastic solid containing a flat annular crack under torsion. Bulletin of JSME, 19, 134, pp. 857-862.

Shibuya, T., Nakahara, I. and Koizumi, T. (1975) The axisymmetric distribution of stresses in an infinite elastic solid containing a flat annular crack under internal pressure. ZAMM, 55, pp. 395-402.