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THERMO-MECHANICAL ANALYSIS OF A FUNCTIONALLY GRADED ANNULAR FIN

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Abstract: *Temperature distribution and thermal stress analyses are presented for an annular fin of functionally graded material (FGM). The closed form solution of stress field is obtained by solving the steady-state nonlinear differential equation of heat transfer using homotopy perturbation method (HPM) coupled with elasticity theory. The properties of the fin material are expressed as linear and power law distribution with temperature and radius. The effect of various thermal parameters on the temperature field, and subsequently stress field are discussed.*

Keywords: Functionally graded fin, Homotopy perturbation method, Inverse analysis, Thermal stresses.

1. Introduction

It is common observation and experience that the stresses are induced in the fin material due to nonuniform temperature distribution. Thus, a significant concentration on thermal stress analysis is required for fin designing. Nonlinear heat transfer analyses for isotropic annular fins have been well studied (Kraus et al., 2001 and Ganji et al., 2011), but there are few works on thermal stress analysis. The thermal stress analysis for a perfect elastic isotropic circular fin was presented by Chiu et al. (2002) using Adomian's double decomposition method. Lee et al. (2002) employed Laplace transformation coupled with finite difference method for thermo-elastic analysis of an annular fin. Mallick et al. (2015) recently presented closed form solution for thermal stresses of an isotropic annular fin using HPM coupled with classical thermo-elasticity equation. An open literature search reveals that no one attempt to study the fin of functionally graded material (FGM) subjected to thermal loading.

This work presents thermal stress analysis for FGM annular fin. The nonlinear heat transfer equation has been solved using HPM. The stress fields are obtained from temperature field, coupled with the elasticity equation. The HPM solution are compared with the results obtained using finite difference method.

2. Governing equations

2.1. Temperature distribution

Let us consider an annular fin made of functionally graded material as shown in Fig. 1. The base temperature of the fin is assumed to be a constant temperature, T_b and its tip is considered to be adiabatic.

The energy balance equation for heat transfer together with boundary conditions can be expressed as (Kraus et al., 2001):

$$t\frac{d}{dr}\left[\left\{k(T)+k(r)\right\}r.\frac{dT}{dr}\right]-2h(T)r(T-T_a)-2\varepsilon\sigma r(T^4-T_s^{\,4})+q(T)tr=0 \quad (1)$$

$$T=T_b \text{ at } r=r_i \text{ and } \frac{dT}{dr}=0 \text{ at } r=r_o \quad (2)$$

where $k(T)=k_o\left\{1+\kappa\left(T-T_a\right)\right\}$, $k(r)=k_o\left\{1+\gamma\left(r-r_i\right)/\left(r_o-r_i\right)\right\}$, $q(T)=q_o\left\{1+e\left(T-T_a\right)\right\}$, and

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$h(T) = h_b \left\{ (T - T_a) / (T_b - T_a) \right\}^n$.

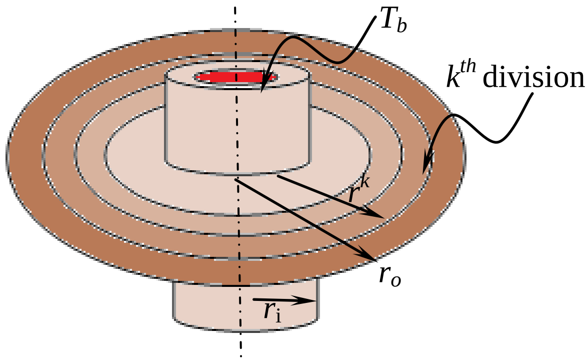

Fig. 1: Fin geometry of FGM.

Introducing the non-dimensional parameters, the energy balance equation and the boundary conditions are reduced to:

$$\theta'' + \frac{\beta}{2}(\theta - \theta_a)\theta'' + \frac{\gamma\xi}{2(R-1)}\theta'' + \frac{\beta}{2}(\theta')^2 + \frac{\gamma}{2(R-1)}\theta' + \frac{1}{2(1+\xi)}\theta' + \frac{\beta}{2}\frac{(\theta - \theta_a)}{1+\xi}\theta' + \frac{\gamma}{2(R-1)}\frac{\xi}{1+\xi}\theta' - N_c\frac{(\theta - \theta_a)^{(m+1)}}{(1-\theta_a)^m} - N_r(\theta^4 - \theta_s^4) + \frac{G}{2}\left\{1 + E_G(\theta - \theta_a)\right\} = 0 \quad (3)$$

$$\theta = 1 \text{ at } \xi = 0 \text{ and } \theta' = 0 \text{ at } \xi = R - 1 \quad (4)$$

All the above terms are described separately in the nomenclature. Constructing the HPM formulation, Eq. 3 can be rewritten as,

$$(1-p)L(\theta - \theta_0) + p\left(\theta'' + \frac{\beta}{2}(\theta - \theta_a)\theta'' + \frac{\gamma\xi}{2(R-1)}\theta'' + \frac{\beta}{2}(\theta')^2 + \frac{\gamma}{2(R-1)}\theta' + \frac{1}{2(1+\xi)}\theta' + \frac{\beta(\theta - \theta_a)}{2(1+\xi)}\theta' + \frac{\gamma\xi}{2(R-1)(1+\xi)}\theta' - N_c\frac{(\theta - \theta_a)^{(m+1)}}{(1-\theta_a)^m} - N_r(\theta^4 - \theta_s^4) + \frac{G}{2}\left\{1 + E_G(\theta - \theta_a)\right\}\right) = 0 \quad (5)$$

where $p \in [0, 1]$ is an imbedding parameter and L denotes the linear operator as $d^2/d\xi^2$. The HPM solution converges for $p = 1$, and the final solution for temperature field can be expressed as:

$$\theta = \theta_o + \theta_1 + \theta_2 + \ldots.. \quad (6)$$

The details pertaining to the solution procedure have been presented in ref. (Mallick et al., 2015).

2.2. Thermo-elastic solution

For functionally graded material, the elastic constant and the co-efficient of thermal expansion are assumed to be a function of fin radius:

$$E(r) = E_0 (r/r_i)^{n_1} \text{ and } \alpha(r) = \alpha_0 (r/r_i)^{n_2} \quad (7)$$

Introducing a stress function, $\psi(r)$, such that $\sigma_r = \psi / r$ and $\sigma_\phi = d\psi / dr$ which satisfy the stress equation of equilibrium. Applying Hooke's law for plane stress condition and employing the strain compatibility condition, the differential equation in terms of stress function in the non-dimensional form yields,

$$\xi_1^2 \frac{d^2\psi}{d\xi_1^2} + (1-n_1)\xi_1 \frac{d\psi}{d\xi_1} + (\nu n_1 - 1)\psi = -Er_i\chi\xi_1^{(n_1+n_2+2)}\frac{d\theta}{d\xi_1} - n_2 Er_i\chi\xi_1^{(n_1+n_2+1)}\theta \quad (8)$$

where ξ_1 (= r/r_i) is the non-dimensional radius. The general solution for $\psi(\xi_1)$ is given as,

$$\psi = D_1\xi_1^{\eta_1} + D_2\xi_1^{\eta_2} + A_0\xi_1^{(\eta_3+1)} + A_1\xi_1^{(\eta_3+2)} + A_2\xi_1^{(\eta_3+3)} + A_3\xi_1^{(\eta_3+4)} + A_4\xi_1^{(\eta_3+5)} + B\xi_1^{(\eta_3+2)}\log\xi_1 \quad (9)$$

The constants D_1 and D_2 are obtained from the boundary conditions, $\sigma_r = 0$ at bore and tip of the fin.

The terms $\eta_{1,2} = n_1/2 \mp \sqrt{(1-\nu n_1 + n_1^2/4)}$,

$$A_0 = \frac{4n_2\chi L_0}{\left\{\left(n_1^2 - 4n_1\nu + 4\right) - \left(n_1 + 2n_2 + 2\right)^2\right\}},$$

$$A_1 = \frac{4\chi\left\{\left(1+n_2\right)L_1 + L_5\right\}}{\left\{\left(n_1^2 - 4n_1\nu + 4\right) - \left(n_1 + 2n_2 + 4\right)^2\right\}},$$

$$A_4 = \frac{4\chi\left(4+n_2\right)L_4}{\left\{\left(n_1^2 - 4n_1\nu + 4\right) - \left(n_1 + 2n_2 + 10\right)^2\right\}}$$ and

$$B = \frac{16\chi\left(1+n_2\right)L_5}{\left\{\left(n_1^2 - 4n_1\nu + 4\right) - \left(n_1 + 2n_2 + 4\right)^2\right\}^2}.$$

3. Results and discussion

The closed form solution and inverse study of an axisymmetric annular fin of functionally graded material is presented. Unless mentioned otherwise, the values of non-dimensional *parameters* $N_c = 0.5$, $N_r = 0.2$, $\beta = 0.2$, $m = 0.25$, $\gamma = 0.2$, $G = 0.2$, $n_1 = 0.5$, $n_2 = 0.5$ and $R = 2$ are to be taken in the analysis. Fig. 2 shows the HPM results for the temperature field. For the correctness of the solution, the temperature field has been compared with finite difference solution. The result shows only 2.7 % variation in the temperature at the tip. The effect of various thermal parameters, N_c, N_r, β and γ, on the temperature field are presented in Fig. 3. It can be seen that the local temperature along the fin radius gradually decreases for all the cases. With the increase of the parameters β *and* γ, responsible for the variation of thermal conductivity, the local temperature field increases. This result suggests that the heat transfer through the fin material enhances with the increasing of variable thermal conductivity parameters. On the other side, it can be noticed that the heat transfer process expedites with decrease in parameters, N_c and N_r. The variation of radial and tangential stresses developed due to the variation of temperature gradient along the fin radius. The effect of the parameters describing the variation of Young's modulus (n_1) and coefficient of thermal expansion (n_2) on the thermal stresses are presented in Fig. 4 and Fig. 5. It can be seen, the magnitude of radial and tangential stresses increase with the increase of parameter, n_1, as the elastic modulus increases with n_1. The tangential stress near to the bore is found to be compressive and tensile is near to the tip. On the other hand, for the positive value of n_2, the co-efficient of thermal expansion exponentially increases from the base to tip of the fin. As a result, the tendency of tensile behaviour in radial stress is predominant when the value of n_2 is positive. Just opposite behaviour is observed when n_2 is negative.

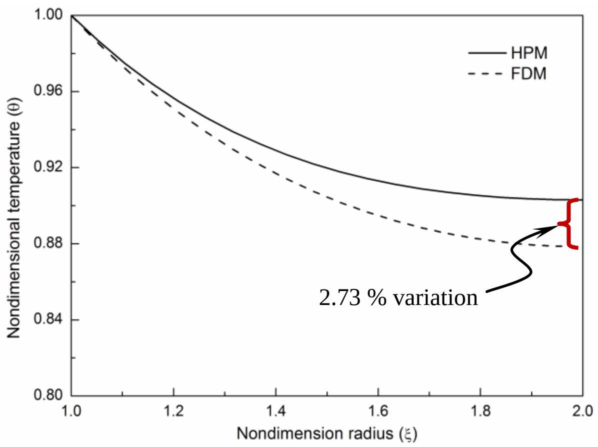

Fig. 2: Validation of HPM solution for temperature distribution in a fin.

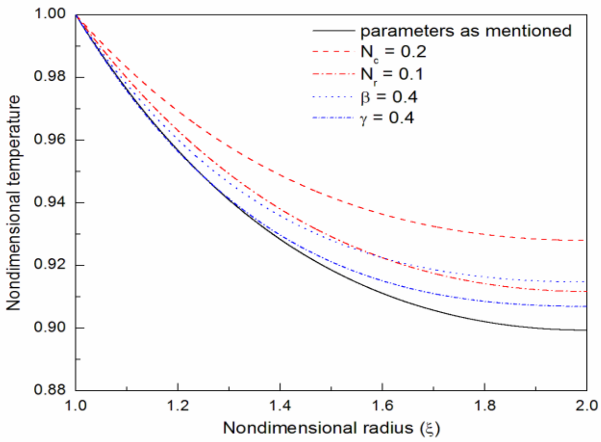

Fig. 3: Effect of various thermal parameters on the variation of temperature distribution.

4. Conclusions

An approximate closed form solution for temperature and stress field in a fin of FGM has been derived successfully. The present work gives an open choice to the designer for selecting and adjusting the fin parameters for a desired temperature and stress fields. The stress field mainly depends on the temperature gradient, as well as, the parameters responsible for the variation of Young's modulus and the coefficient of thermal expansion. Mainly, the tangential stress dominates in the fin material and may be responsible for the mechanical failures due to thermal loading.

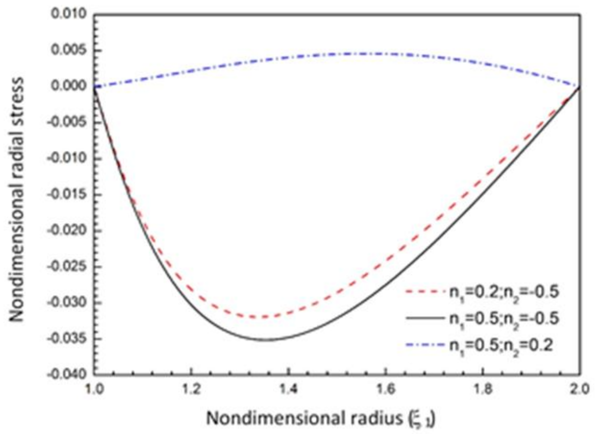

Fig. 4: Effect of the parameters n_1 and n_2 on radial stresses.

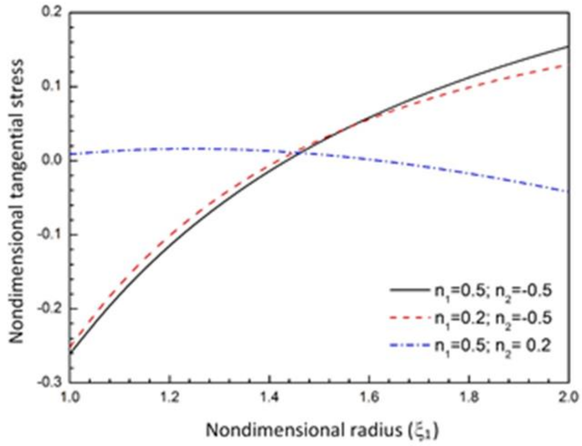

Fig. 5: Effect of the parameters n_1 and n_2 on tangential stresses.

Nomenclature

r_i, r_o, t	inner radius, outer radius and thickness of the fin,
$h(T)$, $k(T)$, $k(r)$	variable heat transfer coefficient and thermal conductivity parameter,
h_b, q_o, ε	heat transfer coefficient, heat generation and emissivity parameters,
k_0	thermal conductivity at convection sink temperature or at the bore of the fin,
κ, γ, λ, e	parameters describing the linear variation of thermal conductivity, surface emissivity and internal heat generation,
β	non-dimensional parameter describing the variation of thermal conductivity with respect to temperature, $\beta = \kappa T_b$,
m	exponent of variable convective heat transfer coefficient,
N_c	non-dimensional thermo-geometric parameter, $(2hr_i^2/k_o t)^{0.5}$,
N_r	non-dimensional conduction-radiation parameter, $(2r_i^2\sigma\varepsilon_s T_b^3/k_o t)$,
G	non-dimensional heat generation parameter, $G = q_o r_i^2/k_o T_b$,
E_G	non-dimensional parameter describing the variation of heat generation, $E_G = eT_b$,
T_b, T_a, T_s	base temperature of fin, ambient temperature and radiation sink temperature,
θ	dimensionless temperature, and dimensionless radiation sink temperature, $\theta = T/T_b$,
θ_a, θ_s	dimensionless convection and radiation sink temperature, $\theta_a = T_a/T_b$ and $\theta_s = T_s/T_b$,
ξ, ξ_1	dimensionless radius of fin, $\xi = (r - r_i)/r_i$ and $\xi_1 = r/r_i$,
R	annular ratio, $R = r_o/r_i$,
$\bar{\sigma}_r$ $\bar{\sigma}_\theta$	non-dimensional radial stress ($\bar{\sigma}_r = \sigma_r/E_0$) and tangential stress ($\bar{\sigma}_\phi = \sigma_\phi/E_0$),
$E(r)$, $\alpha(r)$	variation of elastic modulus and co-efficient of thermal expansion,
n_1, n_2	power index of elastic modulus and co-efficient of thermal expansion variation,
E_0, α_0	modulus of elasticity and co-efficient of thermal expansion at the base of the fin,
χ	non-dimensional coefficient of thermal expansion, $\chi = \alpha_0 T_b$.

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