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PARAMETRIC INSTABILITY OF THE PLANE FREE SURFACE OF THE LIQUID IN CYLINDRICAL STORAGE TANKS

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Abstract: *The paper deals with the dynamic analysis of the ground-supported cylindrical vertical liquid storage tanks with the aim to determine the response of the upper liquid portion during vertical excitation. Tanks are used as storage for liquids in various sectors of industry. Hence, it is a request for satisfactory performance during dynamic loadings (e.g. earthquakes). The paper is dedicated to the phenomenon associated with the vertical oscillations of the liquid which results in the standing waves at the free surface known as Faraday waves. The analysis of a linear mathematical model for ideal liquid subjected to the vertical excitation with the constant amplitude and frequency is presented. It takes into account the theory introduced by Benjamin and Ursell which leads to the Mathieu equation for parametric vibration. Using Mathieu formulation, the mth mode of oscillation is excited when the combination of parameters corresponding to the amplitude and frequency lies in an unstable region of the stability chart. The free surface remains plane when assuming a pair of parameters in a stable region.*

Keywords: Faraday waves, Parametric instability, Mathieu equation, Free surface, Liquid storage tank.

1. Introduction

Tanks containing liquid are used as storage in various industry sectors. Their ordinary operation can be threatened by loading of various nature which may result in negative consequences. Dynamically loaded tank-liquid systems can take a variety of damages to which are exposed and which are caused by the oscillation of the liquid. There are procedures applied for evaluating dynamic effects in liquid storage tanks. The most widely used is the one based on a spring-mass model (Housner, 1954) in which the total liquid mass is divided into two zones – impulsive and convective. The impulsive zone is a part representing the effects of the portion of liquid which moves in unison with the tank. The convective part represents the free surface which moves against the walls. Using Housner's theory, this zone of liquid can be substituted for an infinite number of convective masses connected to the tank with springs of appropriate stiffness. Each of the masses represents another mode of oscillation of the convective liquid. In addition to the spring-mass model, the convective liquid can be replaced by another equivalent – system of simple pendulums (Ibrahim, 2005). For a formulation of specific equations describing a behavior of the convective liquid, classical Euler hydrodynamics theory is used instead of simplified models (scientific works of Faraday, Rayleigh, Matthiessen, etc.).

One of the phenomena observed at the free surface is a sloshing of the liquid which can be caused by parametric excitation. These standing sloshing waves are known as Faraday waves named after Michael Faraday who first observed and described them. Parametric oscillations are the results of having time-varying (periodic) parameters in the system. When the system experiences parametric resonance, the amplitude of the oscillations of the system will be increased exponentially. For parametrically excited sloshing, the effective gravitational field becomes time dependent. Sloshing waves can be generated when the liquid in the vessel is vertically excited at the frequency close to the twice the natural frequency of the convective liquid. Oscillation of the convective liquid in the containers may lead to negative effects such

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as deformations of the tank walls (closed tanks) or liquid spilling (tanks without roofs). Therefore, the sufficient freeboard between the free surface and the top of the tank must be designed.

2. Basic concept of the parametric oscillation of the contained liquid

This section shortly describes the procedure for the parametric oscillation of the free surface subjected to the vertical periodic motion. It is based on the work published by T. B. Benjamin and F. Ursell (1954) which uses the Euler equations of hydrodynamics and the equation of continuity for the formulation of the Mathieu equation of the free surface of liquid. It is assumed a vertical cylindrical model of a vessel of an arbitrary cross-section with horizontal free surface and bottom at depth H_{L} (Fig. 1).

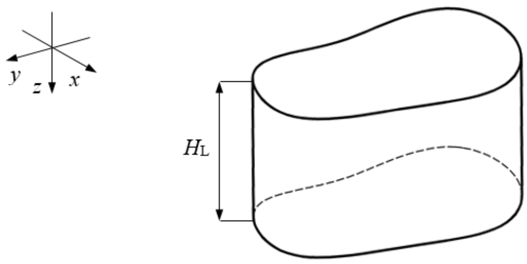

Fig. 1: Model of the tank containing liquid.

The liquid in the vessel is assumed as ideal (i.e. incompressible and non-viscous) due to small viscosity effects which may be neglected. The vessel-liquid system is moving periodically in the vertical z-direction with imparted acceleration of $a \cos \omega t$ in which a represents a maximum acceleration and ω is an angular frequency. The vertical oscillation can be represented by the effective gravitational acceleration $(g - a \cos \omega t)$ while the vessel walls remain at rest. This effective gravitational acceleration is time-varying parameter whose consequence results in a parametric resonance of Faraday waves.

After introducing the Euler equations of hydrodynamics, the equation of continuity and basic assumptions for velocity potential $(u, v, w) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi$, pressure at free surface $p = \gamma\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right)$ and the equation of the free surface $z = \zeta(x, y, t)$, the equation of motion at $z = 0$ can be defined as

$$\frac{\gamma}{\rho}\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) + \left(\frac{\partial \phi}{\partial t}\right)_{z=0} + \frac{1}{2}(u^2 + v^2 + w^2) - (\, g - a \cos \omega t)\zeta = 0 \qquad (1)$$

where ρ represents the constant density during the motion and γ represents the surface tension. For the further formulation of Mathieu equation, the procedure is simplified by assuming that the liquid velocity and displacement of the free surface are sufficiently small. Therefore, (1) is linearized by omitting squares of velocities.

Assuming boundary conditions (Benjamin, 1954), the solution of wave equation $\zeta(x, y, t)$ may be presented as a superposition of particular solutions and they are expressed as the product of two functions, first of which depends only on time and second one depends only on space coordinates.

$$\zeta(x, y, t) = \textstyle\sum_0^\infty a_m(t)\, S_m(x, y) \qquad (2)$$

Applying the substitution of (2) into wave equation, Helmholtz equation can be obtained

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_m^2\right) S_m(x, y) = 0 \qquad (3)$$

where k_m represents the wave number. The pressure at the free surface and the potential can be developed using particular solutions respectively

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} = -\textstyle\sum_0^\infty k_m^2\, a_m(t)\, S_m(x, y) \qquad (4)$$

$$\phi(x, y, z, t) = -\textstyle\sum_1^\infty \frac{\mathrm{d}a_m(t)}{\mathrm{d}t} \frac{\cosh k_m(H_{\mathrm{L}} - z)}{k_m \sinh k_m H_{\mathrm{L}}}\, S_m(x, y) \qquad (5)$$

Substitution of (2), (4) and (5) into (1) shows that

$$\textstyle\sum_1^\infty \frac{S_m(x,y)}{k_m \tanh k_m H_{\mathrm{L}}} \left[\frac{\mathrm{d}^2 a_m}{\mathrm{d}t^2} + k_m \tanh k_m H_{\mathrm{L}} \left(\frac{k_m^2 \gamma}{\rho} + g - a \cos \omega t\right) a_m\right] = 0 \qquad (6)$$

which must satisfy

$$\frac{\mathrm{d}^2 a_m}{\mathrm{d}t^2} + k_m \tanh k_m H_\mathrm{L} \left(\frac{k_m^2 \gamma}{\rho} + g - a\cos\omega t\right) a_m = 0 \tag{7}$$

The parameters P_m and Q_m are introduced using (7)

$$P_m = \frac{4\, k_m \tanh k_m H_\mathrm{L}}{\omega^2}\left(g + \frac{k_m^2 \gamma}{\rho}\right) \quad Q_m = \frac{2\, k_m a \tanh k_m H_\mathrm{L}}{\omega^2} \tag{8}$$

and after substitution $\tau = \frac{1}{2}\omega t$ and assuming parameters defined in (8), the equation (9) represents the standard form of the Mathieu equation

$$\frac{\mathrm{d}^2 a_m}{\mathrm{d}\tau^2} + (P_m - 2Q_m \cos 2\tau) a_m = 0 \tag{9}$$

Mathieu equation can have solutions depending on the values of the parameter P_m and Q_m. The stability chart (Fig. 2a) reflects regions in which for the point (P_m, Q_m) the solution is stable, i.e. bounded (white regions) or unstable, i.e. unbounded with the time (shaded regions).

3. Mode of oscillation of the liquid

The following section is focused on the analysis of the parametric oscillation of the liquid in the circular vertical tank of the radius *R* (0.2 m), the height *H* (0.32 m) and the wall thickness *t* (5e-4 m). The vessel is filled with liquid (water) to the depth H_L (0.24 m).

The aim is to determine the mode of oscillation (1, 2) and respective regions of instability in dependence on the frequency and the amplitude of the acceleration. From the formulation of the Mathieu equation, it was shown, values P_m and Q_m depend on the wave number. Since modes have different wave numbers, it follows each of them has different stability chart of excitation frequency vs. amplitude of acceleration.

Using (8), frequency and amplitude of acceleration are expressed as

$$f = \frac{1}{\pi}\sqrt{\frac{k_m \tanh k_m H_\mathrm{L}}{P_m}\left(g + \frac{k_m^2 \gamma}{\rho}\right)} \quad a = \frac{2\, Q_m \left(g + \frac{k_m^2 \gamma}{\rho}\right)}{P_m} \tag{10}$$

Applying the values from the boundaries of stability chart (Fig. 2a) into (10), assuming the (1, 2) mode of oscillation and its wave number equal to 5.33 / *R*, the stability chart is defined and presented in Fig. 2b.

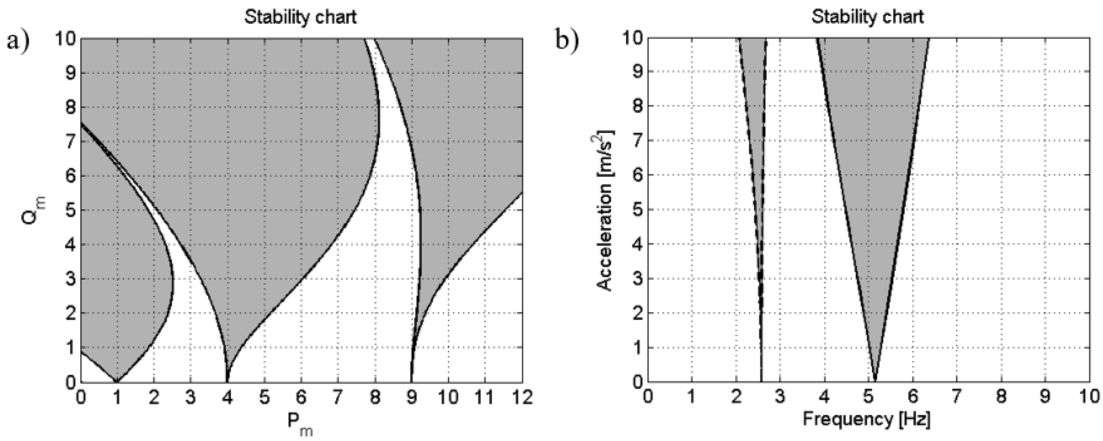

Fig. 2: Stability chart for the (1, 2) mode of oscillation.

If the combination of the excitation frequency and the amplitude of acceleration lies in any of the white regions (Fig. 2b), the solutions of the Mathieu equation can be stable with an oscillatory periodic solution. For other combinations lying in any of the shaded regions, the solutions are unstable with oscillatory exponentially increasing amplitude. In the Fig. 2b, a dashed and solid instability boundaries are depicted. The former represents the region of the (isochronous) mode with the frequency equal to the excitation frequency; the latter reflects the region for the mode with half frequency of the excitation.

Using the analytical procedure for definition of the (1, 2) mode shape (Fig. 3a) considering the investigated circular tank of radius *R* and using polar coordinates in terms of which S_m must satisfy

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+k_m^2\right)S_m=0 \tag{11}$$

$$S_{l,m}=J_l(k_{l,m}r)\cos l\theta \tag{12}$$

where J_l represents the Bessel function and $k_{l,m}$ is the mth zero of $J_l'(k_{l,m}R)$.

The natural frequency of the mode shape with respective wave number may be calculated using following equation

$$f_{l,m}=\frac{1}{2\pi}\sqrt{\tanh k_{l,m}H_{\mathrm{L}}\left(\frac{k_{l,m}^3\gamma}{\rho}+k_{l,m}g\right)} \tag{13}$$

From the analysis, it can be observed the (1, 2) mode occurs at excitation principal frequency 5.16 Hz. Applying (13), the natural frequency of the respective mode of oscillation is at 2.58 Hz. Using finite element analysis in ANSYS Multiphysics this phenomenon occurs at 2.61 Hz and the mode is presented in Fig. 3b. Results between each solution represent good conformity.

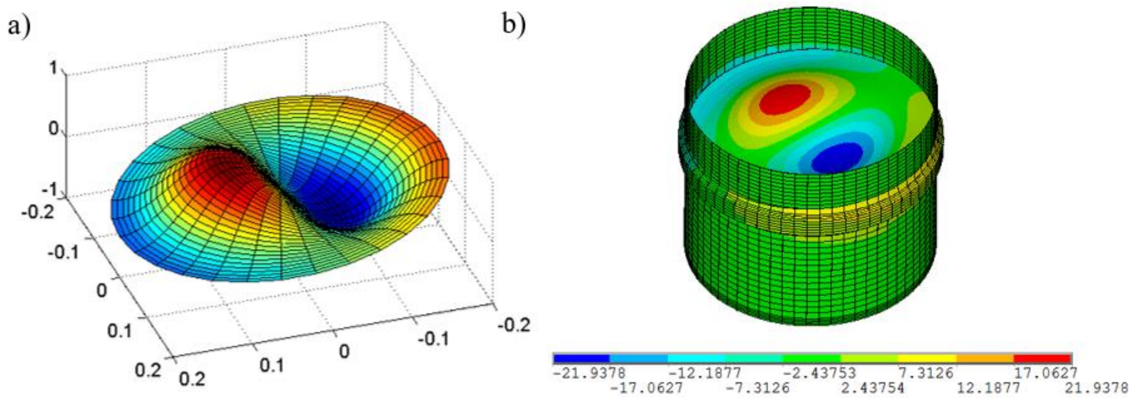

Fig. 3: (1, 2) mode of oscillation using analytical and numerical computation.

4. Conclusions

The aim of this paper was focused on the parametrically excited sloshing waves at the free surface due to the time-varying effective gravitational acceleration. Mathieu equation of the free surface was formulated using an ideal liquid theory. Stable or unstable solutions of the equation depend on the combinations of P_m and Q_m values representing the location in the stability chart. At the parametric resonance, the amplitude of liquid oscillation is exponentially increasing. However, for real liquids, the amplitudes have limited values due to damping and nonlinear effects (not included in the ideal liquid theory). Since modes have different wave numbers, each of them has different stability chart of excitation frequency vs. amplitude of acceleration. Following results, it can be said that isochronous modes have narrower unstable regions than modes with half excitation frequency. Therefore, modes at this frequency are more difficult to excite and observe experimentally. To sum up, when investigating unstable solutions of the individual modes, each of them has the respective frequency at which a non-viscous liquid can be excited by an arbitrarily small amplitude of acceleration. But in a real liquid, a limit amplitude must be exceeded. It is affected by various factors such as viscosity, tank geometry and liquid depth.

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