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ON THE ENERGY RELEASE RATE OF THE CRACK EMANATING FROM THE INCLUSION INTERPHASE

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Abstract: *The problem of the crack emanating from the interphase region of the circular inclusion is investigated. The problem combines an application of dislocation distribution technique for a crack modelling and the method of boundary integral equations to approximate the loading along the boundary of the domain containing an inclusion. The topological derivative method provides the combination of both approaches and results to the evaluation of the energy release rate of the arbitrary oriented microcrack emanating from the inclusion and matrix interphase. The fundamental solution intended to the boundary integral method such as the continuously distributed dislocation technique is based on the application of Muschelishvili complex potentials in the form of the Laurent series. The coefficients of the series are evaluated from the compatibility conditions along the interfaces of inclusion, interface and matrix.*

Keywords: **Microcrack, Inclusion, Interphase, Singular integral equation, Dislocation distribution technique, Complex potentials.**

1. Introduction

The paper takes on the scheme of the energy release rate G associated with a finite small crack initiation at any boundary location based on the topological derivative method, Silva et al. (2011). It is supposed that the crack is initiated and emanating from the interface region between the inclusion and arbitrarily loaded finite matrix in the studied problem. The crack initiation from some tip of the sharp shaped inclusion can be appeared in the silicate-based composites. The applied simplified mathematical model can be used to study the influence of the material mismatch of the inclusion/matrix interphase to the fracture toughness of this composite even that the mathematical simplification suppresses the sharpness of the inclusion. The topological derivative field indicates the variation of a response functional when an infinitesimal hole is introduced in the body. The response functional is the total potential energy in the discussed problem. The kernel of the mathematical scheme is the fundamental solution of the unit point force or edge dislocation interacting with an inclusion and an interphase, Cheeseman et al. (2001). This fundamental solution is applied to the approximation of the loaded boundary of the finite uncracked matrix containing an inclusion such as to model the crack. There is many ways to model the uncracked matrix with the inclusion and its interphase. The most common tool is the finite element method, but due to the modesty of the external domain shape and the emphasis on the influence of the inclusion interphase on the near stress distribution along the inclusion in the matrix, the boundary integral method was used. The boundary integral method allows one to rather precise evaluate the stress and strain distribution in the matrix near the inclusion interphase independently of the quality of the domain and especially thin interphase mesh. The second application of the bellow mentioned fundamental solution is the crack model via the continuously distributed dislocation technique, Hills et al. (1996). The condition that the crack faces have to be stress free, the continuously distributed dislocations technique leads to the solution of the integral equation with the Cauchy type singular kernel. The advantage of this method is the simply evaluation of the stress intensity factor of the crack as the limiting value of Jacobi polynomials which are

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used to interpolate the continuous array of dislocations along the crack. The asymptotic analysis combines both solutions, i.e. the non-cracked finite domain and cracked infinite domain. It is based on the stress composite expansion, evaluating of the energy momentum tensor and approximation of the energy release rate for any crack size by means of topological derivative, Silva et al. (2011).

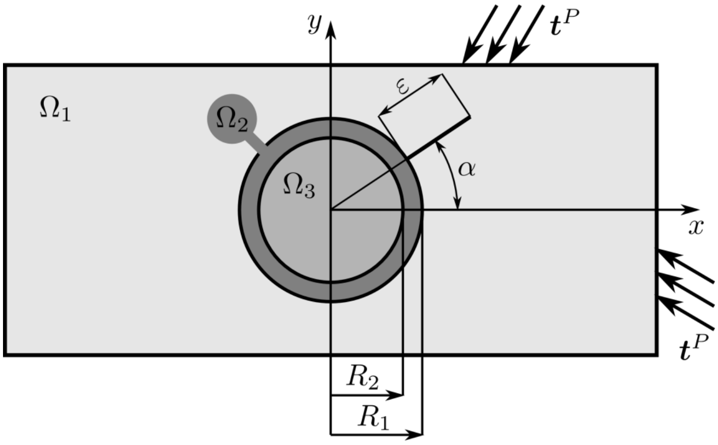

Fig. 1: The geometry of the studied problem.

2. Topological derivative for cracked body

It is known that the shape sensitivity of the total potential energy ψ with respect to crack length ε is given by the energy release rate, Silva et al. (2011),

$$G(\Omega_\varepsilon) = \frac{d}{d\varepsilon}\psi(\Omega_\varepsilon), \tag{1}$$

where the derivative is meant in the shape sense and Ω_ε is the domain with a small crack in the matrix with boundary $\partial\Omega_\varepsilon = \partial\Omega \cup \gamma_\varepsilon$ with γ_ε being the crack boundary and $\partial\Omega = \bigcup_{i=1,2,3}\partial\Omega_i$, where $\partial\Omega_i$ are matrix and interface boundary between matrix or inclusion and their interphase, Fig. 1. Then the total potential energy is given by

$$\psi(\Omega_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon}\nabla^s \boldsymbol{u}_\varepsilon : \boldsymbol{T}_\varepsilon \mathrm{d}V - \int_{\partial\Omega_1}\boldsymbol{t}^P \cdot \boldsymbol{u}_{1\varepsilon}\,\mathrm{d}S -$$

$$- \int_{\partial\Omega_2}(\boldsymbol{T}_{1\varepsilon}\mathbf{n_2}\cdot\boldsymbol{u}_{1\varepsilon} + \boldsymbol{T}_2\mathbf{n_2}\cdot\boldsymbol{u}_2)\,\mathrm{d}S - \int_{\partial\Omega_3}(\boldsymbol{T}_2\mathbf{n_3}\cdot\boldsymbol{u}_2 + \boldsymbol{T}_3\mathbf{n_3}\cdot\boldsymbol{u}_3)\,\mathrm{d}S. \tag{2}$$

In the above the index ε denotes the response quantities evaluated on the crack domain, $\boldsymbol{u}$ is the displacement vector, $\nabla^s\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, $\boldsymbol{T}$ is the symmetric Cauchy stress tensor and tractions $\boldsymbol{t}^P$ are applied to the boundary $\partial\Omega_1$. The displacements and stress tensor satisfying the governing equations of the linear elasticity on the domain Ω_ε,

$$\mathrm{div}\boldsymbol{T}_\varepsilon = 0 \text{ in } \Omega_\varepsilon, \boldsymbol{T}_{1\varepsilon}\mathbf{m} = 0 \text{ on } \gamma_\varepsilon, \boldsymbol{T}_{1\varepsilon}\mathbf{n_1} = \boldsymbol{t}^P \text{ on } \partial\Omega_1, \tag{3}$$

$$\boldsymbol{T}_{1\varepsilon}\mathbf{n_2} = -\boldsymbol{T}_2\mathbf{n_2} \wedge \boldsymbol{u}_{1\varepsilon} = \boldsymbol{u}_2 \text{ on } \partial\Omega_2, \boldsymbol{T}_2\mathbf{n}_3 = -\boldsymbol{T}_3\mathbf{n}_3 \wedge \boldsymbol{u}_2 = \boldsymbol{u}_3 \text{ on } \partial\Omega_3, \tag{4}$$

where $\boldsymbol{m}$ and $\boldsymbol{n}$ are the outward normal to γ_ε and $\partial\Omega$, respectively. The body forces are neglected and the crack faces are assumed to be traction free. The evaluation of the topological derivation (1) leads to the relation, e.g. Silva (2011),

$$J(\boldsymbol{\Sigma}_\varepsilon) = -\frac{d}{d\varepsilon}\psi(\Omega_\varepsilon) = \lim_{r\to 0}\int_{B_r}\boldsymbol{\Sigma}_\varepsilon\mathbf{n}\cdot\boldsymbol{e}_r dS, \tag{5}$$

where B_r is a ball of radius r centered at the crack tip, $\boldsymbol{e}_r$ is a unit vector aligned with the crack and oriented in the crack growth direction and $J(\boldsymbol{\Sigma}_\varepsilon)$ is the J-integral joined with the energy momentum tensor $\boldsymbol{\Sigma}_\varepsilon$.

3. Fundamental solution

The evaluation of the displacement and stress field at the crack tip depends on the knowledge of the fundamental solution, i.e. solution describing the interaction between the unit point force or dislocation and the circular inclusion with thin interphase, see Cheeseman et al. (2001). The problem is formulated using Muskhelishvili complex potentials in the form of Laurent series,

$$\varphi_1(z) = \sum_{k=1}^{\infty} A_k z^k + \sum_{k=1}^{\infty} h_{-k} z^{-k},\ \psi_1(z) = \sum_{k=1}^{\infty} B_k z^k + \sum_{k=1}^{\infty} p_{-k} z^{-k}, \quad (6)$$

$$\varphi_2(z) = \sum_{k=-\infty}^{\infty} q_k z^k,\ \psi_2(z) = \sum_{k=-\infty}^{\infty} r_k z^k, \quad (7)$$

$$\varphi_3(z) = \sum_{k=1}^{\infty} s_k z^k,\ \psi_3(z) = \sum_{k=1}^{\infty} t_k z^k, \quad (8)$$

where the coefficients A_k and B_k are known and account for the point force or dislocation singularity. The coefficients h_{-k}, p_{-k}, q_k, r_k, s_k, t_k are evaluated from the compatibility conditions (4).

4. Asymptotic analysis

Assuming the crack length ε is sufficiently small with respect to the inclusion size R_2 the evaluation of the stress tensor $\boldsymbol{T}_\varepsilon(\boldsymbol{x})$ at the crack tip and after that the energy momentum tensor $\boldsymbol{\Sigma}_\varepsilon$ in (5) can be provided using the following stress composite expansion, Silva et al. (2011),

$$\boldsymbol{T}_\varepsilon(\boldsymbol{x}) = \boldsymbol{T}(\hat{\boldsymbol{x}}) + \widetilde{\boldsymbol{T}}(\boldsymbol{y}) + O(\varepsilon), \quad (9)$$

where $\hat{\boldsymbol{x}} \in \partial\Omega_2$ is reference point of matrix/interphase boundary from which the crack emanates and $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$ is the scaled position vector. The outer stress $\boldsymbol{T}(\hat{\boldsymbol{x}})$ is evaluated along the crack position in the matrix without the crack and under the external load $\boldsymbol{t}^P$. The stress $\boldsymbol{T}(\hat{\boldsymbol{x}})$ is observed from the Hooke's law and the displacement solution of the well-known boundary integral equations appearing in the boundary value problem, Brebbia et al. (1984),

$$c(\hat{\boldsymbol{x}})\boldsymbol{u}(\hat{\boldsymbol{x}}) + \int_{\partial\Omega_1} \boldsymbol{t}^*(\hat{\boldsymbol{x}},\boldsymbol{\xi}) \cdot \boldsymbol{u}(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) = \int_{\partial\Omega_1} \boldsymbol{u}^*(\hat{\boldsymbol{x}},\boldsymbol{\xi}) \cdot \boldsymbol{t}^{\boldsymbol{P}}(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) \quad (10)$$

where the displacements $\boldsymbol{u}^*(\hat{\boldsymbol{x}},\boldsymbol{x})$ and tractions $\boldsymbol{t}^*(\hat{\boldsymbol{x}},\boldsymbol{x})$ follow from the Muskhelishvili complex potentials (6), (7) and (8). They are the response at the point $\boldsymbol{\xi}$ correspond to a unit point force acting at the point $\hat{\boldsymbol{x}}$. The inner stress $\widetilde{\boldsymbol{T}}(\boldsymbol{y})$ annihilates the leading order term of the tractions $\boldsymbol{T}(\hat{\boldsymbol{x}})$ on the crack faces. Because the inner stress is expressed in terms of the stretched position vector $\boldsymbol{y}$, points $\boldsymbol{y}$ far away from the crack correspond to points $\boldsymbol{x}$ only in a small distance from $\hat{\boldsymbol{x}}$. Hence the inner boundary value problem is that of the infinite domain with inclusion and interphase of radius R_2/ε and R_1/ε, respectively, from which emanates the crack of unit length to the matrix. Using the distributed dislocation technique, Hills et al. (1996), the inner stress components $\tilde{T}_{ij}$ are given by,

$$\tilde{T}_{ij}(\boldsymbol{y}) = \int_0^1 \left[b_\theta(t) K_{ij\theta}(\alpha,\boldsymbol{y},t) + b_r(t) K_{ijr}(\alpha,\boldsymbol{y},t)\right] dt, \quad (11)$$

where α is crack orientation, see Fig. 1, and kernels $K_{ij\theta}$ and K_{ijr} are developed from complex potentials (6), (7) and (8). The unknown dislocation densities b_θ and b_r are obtained in such a way, that the negative value of $\boldsymbol{T}(\hat{\boldsymbol{x}})$ is substituted to the left hand side of (11) for $\boldsymbol{y} \in \gamma_1$. Using the polar coordinate system, the inner stress field near the crack tip can be expressed in terms of the Williams asymptotic series,

$$\widetilde{\boldsymbol{T}}(r/\varepsilon,\theta) = \sqrt{\frac{\varepsilon}{r}} A_1 \boldsymbol{F}_1(\theta) + A_2 \boldsymbol{F}_2(\theta) + \sqrt{\frac{r}{\varepsilon}} A_3 \boldsymbol{F}_3(\theta) + O(r). \quad (12)$$

Only the leading term of the expression above contributes to $J(\boldsymbol{\Sigma}_\varepsilon)$. This term is proportional to the squared values of stress intensity factors. Hence, one can get from (1) and (5),

$$G(\varepsilon,\hat{\boldsymbol{x}},\alpha) = \frac{\varepsilon}{E}[K_I^2(\hat{\boldsymbol{x}},\alpha) + K_{II}^2(\hat{\boldsymbol{x}},\alpha)]. \quad (13)$$

5. Numerical example

The numerical examples show the intermediate results of the character of the convergence of the series appearing in the fundamental solution (6), (7) and (8) and its application to the boundary integral method (10) with respect to its dependency on the material properties of the inclusion interphase. Fig. 2a shows the convergence of the displacement u_y of the fundamental solution along the x-axis for various degree n of the Laurent series. The unit force is oriented in the direction of x-axis and it is situated at the point $[30, 30]$ mm. The radius of the inclusion and its interphase are 10 and 11 mm, respectively. The Young modulus of the matrix, interphase and inclusion are 0.8E+11, 0.2E+11 and 0.2E+12 MPa and Poisson's ratios are 0.15, 0.2 and 0.3, respectively. It can be seen that for this randomly chosen values, except the $n = 2$ degree of the Laurent series, the curves cannot be distinguished especially with respect to the FEM solution. Fig. 2b shows the influence of the interphase material on the σ_x stress field in the finite domain under the tension $\sigma_x^P = 100$ MPa. The domain contains an inclusion with the radius 4 mm with 1mm interphase. The width of the domain is 20 mm. The values inside the inclusion are not calculated, but can be evaluated form the received values along the boundary $\partial\Omega_2$. However these values are not necessary for the future analysis. It is interesting to point out the inappropriate values near the outer boundary $\partial\Omega_1$. It is a typical behavior of the boundary integral method near the discretized boundary.

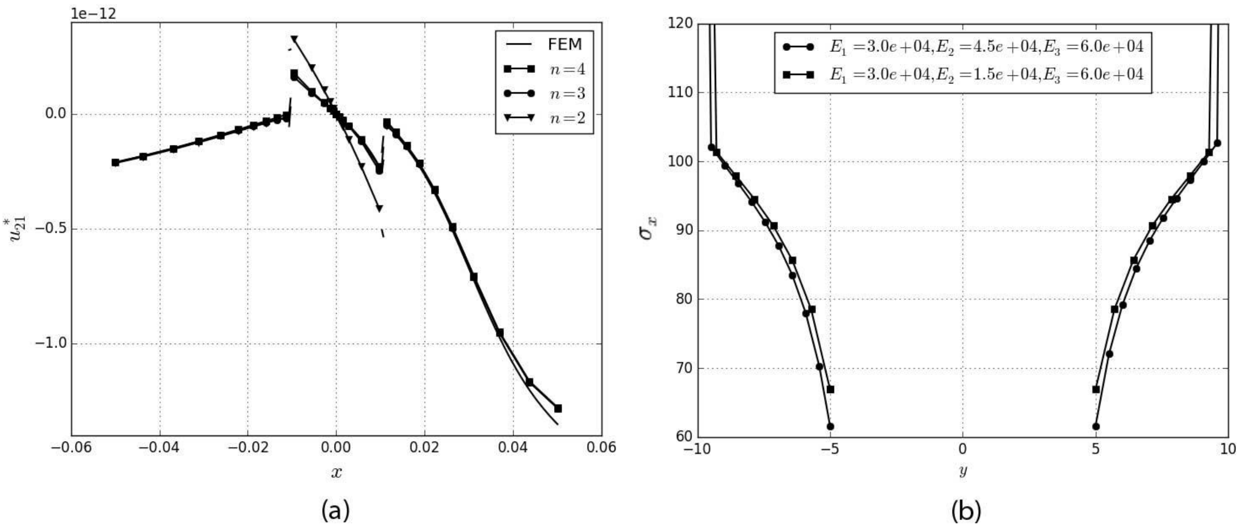

Fig. 2: a) The displacement of the fundamental solution; b) The stress σ_x along the y-axis of the finite domain under the tension $\sigma_x^P = 100$ MPa.

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