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THE ANTIPLANE PROBLEM FOR A STRIP WEAKENED BY A CRACK

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Abstract: *The problem on an arbitrary oriented crack in a strip is solved. The new approach for the elasticity problems with the curvilinear defects is proposed. With the help of the generalized scheme of the integral transformation method the problem is reduced to the singular integral equation, which effective approximate solution can be constructed.*

Keywords: Arbitrary oriented crack, Singular integral equation.

1. Introduction

The problems on the stress concentration near the defects (the cuts, the cracks, the inclusions) play an important role in the modern fracture mechanics. The scheme of the reduction of the elasticity boundary valued problems with a defect to the integral equations, based on the integrals transformation, is well known. In this scheme the defects are inscribed in a certain coordinate system, apparatus of integral transformation is used or perpendicular or parallel to the defect. In proposed paper this methodic is generalized on a case of a curvilinear defect on the example of the antiplane problem for a strip, weakened by an arbitrary oriented crack. It is shown that the problem is reduced to the known singular equation allowing the effective approximate solving.

2. The problem's statement

Let's consider the antiplane problem for a strip $\{ |x|< \infty , |y| < l \}$, the edges of it are fixed. The crack $\{ y = \varphi(x), |x| < 1 \}$ is situated inside the strip. The load of intensity $T(x)$ is applied to the branches of the crack along axis OZ. It is supposed that parameters of crack are such that it is situated inside the strip and don't go to the bound. Without loss of generality (it is required only that function $\varphi(x)$ should be continuous and differentiating on the segment $[-1, 1]$) the final formulas we give for the case of straight crack $\varphi(x) = kx + c$. It is necessary to estimate the stress intensity factor near the crack's ends. The mathematics statement of the formulated problem is following one:

$$\begin{cases} \dfrac{\partial^2 W}{\partial x^2} + \dfrac{\partial^2 W}{\partial y^2} = 0, |x| < \infty, \ y \in (-l, \varphi(x) - 0) \cup (\varphi(x) + 0, l) \\ W, \left.\dfrac{\partial W}{\partial x}\right|_{x\to\infty} \to 0 \\ W|_{y=-l} = W|_{y=l} = 0 \\ \sqrt{1+(\varphi'(x))^2} \left.\dfrac{\partial W}{\partial v}\right|_{y=\varphi(x)-0} = T(x), \quad |x|<1 \end{cases} \tag{1}$$

Here $W(x, y)$ is displacement of point with the coordinates (x, y) along axis OZ. The displacements is continuously different table till the bound of given area, except the angle points – ends of the crack, where $W(x, y)$ should be continuous.

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Let's designate ν as the normal direction to the curve $y = \varphi(x)$ and s as the tangent direction to this curve ν, α is an angle between lines OX and OS. One can get that $tg(\alpha) = \varphi'(x)$ and

$$\begin{cases} \frac{\partial W}{\partial s} = [1 + (\varphi'(x))^2]^{-1}[\frac{\partial W}{\partial x} + \varphi'(x)\frac{\partial W}{\partial y}] \\ \frac{\partial W}{\partial \nu} = [1 + (\varphi'(x))^2]^{-1}[-\varphi'(x)\frac{\partial W}{\partial y} + \frac{\partial W}{\partial y}] \end{cases} \quad (2)$$

Let's note, that if a crack is the straight one and is parallel to one of the coordinate axis, then with the generalized scheme of the integral transformation method the problem will be reduced to a singular integral equation. This equation can be solved and one can get the effective numerical solution. At the case of the arbitrary oriented curvilinear crack the additional difficulties are appeared. They are connected with the changing of the differentiation and integration operations' order at the integral

$$\int_{-l}^{l} \frac{\partial W}{\partial x} \sin \lambda y \, dy$$

when function $W(x, y)$ is the discontinuous one. This work is dedicated to the overcoming of these difficulties.

3. Reduction to the one dimensional problem

Let's apply to the boundary problem (1) the finite sin-Fourier transformation with regard to variable y

$$W_\lambda(x) = \int_{-l}^{l} w(x, y) sin\, \lambda(y + l) dy\,;$$

$$W(x, y) = \frac{1}{l} \sum_{k=1}^{\infty} W_{\lambda_k}(x) \sin \lambda_k (y + l)$$

$$\lambda_k = \frac{k\pi}{2l}$$

Previously one should change the last condition in (1) on the next one

$$<\frac{\partial W}{\partial \nu}>\Big|_{y=\varphi(x)} = \frac{\partial W}{\partial \nu}(x, \varphi(x) - 0) - \frac{\partial W}{\partial \nu}(x, \varphi(x) + 0) = 0$$

$$<W>|_{y=\varphi(x)} = W(x, \varphi(x) - 0) - W(x, \varphi(x) + 0) = \chi(x) \quad (3)$$

where $\chi(x)$ is the unknown function, describing the jump of the function $W(x, y)$ during the transition across the crack. Than $\chi(x) \equiv 0$ when $|x| > 1$. Let's use the integrals:

$$\int_{-l}^{l} \frac{\partial^2 W}{\partial y^2} \sin \lambda y \, dy = <\frac{\partial W}{\partial y}>\Big|_{y=\varphi(x)} \sin(\lambda\varphi) - \chi(x)\lambda \cos(\lambda\varphi) - \lambda^2 W_\lambda(x)$$

$$\int_{-l}^{l} \frac{\partial^2 W}{\partial x^2} \sin \lambda y \, dy = -\varphi(x) <\frac{\partial W}{\partial x}>\Big|_{y=\varphi(x)} \sin(\lambda\varphi) - \frac{d}{dx}[\varphi'(x)\chi(x)\lambda \sin(\lambda x)] + W_\lambda''(x)$$

With regard to the condition (3) one get the one-dimensional boundary problem

$$\begin{cases} \left(\frac{d^2}{dx^2} - \lambda^2\right) W_\lambda(x) = \mathcal{F}_\lambda(x), |x| < \infty \\ W_\lambda, W_\lambda'|_{x\to\mp\infty} \to 0, \mathcal{F}_\lambda(x) = \chi(x)\lambda \cos(\lambda\varphi) + \frac{d}{dx}[\varphi'(x)\chi(x)\lambda \sin(\lambda x)] \end{cases}$$

Its solution can be written in the form:

$$W_\lambda(x) = \int_{-\infty}^{+\infty} \mathcal{F}_\lambda(\xi) \mathrm{G}_\lambda(x,\xi)\, d\xi, \quad \mathrm{G}_\lambda(x,\xi) = -\frac{1}{2\lambda} e^{-\lambda|x-\xi|}$$

$$W_\lambda(x) = \int_{-1}^{1} \chi(\xi)[\lambda \cos(\lambda\, \varphi(\xi))\mathrm{G}_\lambda(x,\xi)\, d\xi - \varphi'(\xi)\sin(\varphi(\xi))\frac{\partial G_\lambda}{\partial \xi}(x,\xi)]d\xi$$

After application of the inverse integral Fourier's transformation one derives

$$W(x,y) = \int_{-1}^{1} \chi(\xi)\frac{\partial G_\lambda}{\partial v_\xi}\big(x,y,\xi,\varphi(\xi)\big)\sqrt{1+[\varphi'(\xi)]^2}d\xi \qquad (4)$$

where derivative $\frac{\partial}{\partial v_\xi}$ is defined by the second formula in (2) after variable changing of the variables (x,y) on the variables (ξ,η).

4. The weak convergent part extraction

The function $\mathrm{G}(x,y,\xi,\eta)$ has form

$$\begin{cases} \mathrm{G}(x,y,\xi,\eta) = \dfrac{1}{l}\displaystyle\sum_{m=1}^{\infty}\left(-\frac{1}{2\lambda_m}e^{-\lambda_m|x-\xi|}\right)\sin\lambda_m(y+l)\sin\lambda_m(\eta+l) \\ \lambda_m = \dfrac{m\pi}{2l} \end{cases}$$

Function $\mathrm{G}(x,y,\xi,\eta)$ can be expressed through the elementary functions with the help of formula (Dwight, 1961)

$$\sum_{m=1}^{\infty}\frac{1}{m}e^{-mA}\sin(mB)\sin(mC) = \frac{1}{4}\ln\frac{sh^2\left(\frac{A}{2}\right)+\sin^2\frac{B-C}{2}}{sh^2\left(\frac{A}{2}\right)+\sin^2\frac{B+C}{2}}$$

$$\mathrm{G}(x,y,\xi,\eta) = \Phi(x-\xi,y-\eta) + \mathrm{G}^*(x,y,\xi,\eta)$$

$$\mathrm{G}^*(x,y,\xi,\eta) = \psi(x-\xi,y-\eta) + \Omega(x-\xi,y+\eta+2l) \qquad (5)$$

$$\Psi(x,y) = \frac{1}{4\pi}\ln[ch(\beta x)-\cos(\beta y)];\ \Phi(x,y) = \frac{1}{2\pi}ln\sqrt{x^2+y^2}$$

$$\Omega(x,y) = -\Psi(x,y) - \Phi(x,y)$$

Let's note that the formula (4) satisfies all conditions of the boundary value problem (1) except last one for any values of the function $\chi(\xi)$. After it one will derive the integral equation on the finite interval (-1, 1) with regard to the unknown function $\chi(x)$

$$\lim_{y\to\varphi(x)\neq 0}\int_{-1}^{1}\chi(\xi)\frac{\partial^2 G}{\partial v_x \partial v_\xi}(x,y,\xi,\eta)\sqrt{1+[\varphi'(x)]^2}d\xi = \sqrt{1+[\varphi'(x)]^2}T(x), |x|<1 \quad (6)$$

All transformations were done for the curvilinear crack $y=\varphi(x)$. To simplify the calculations let's take $y = kx + b$. As a result, after extraction of the singular kernel, one will derive the equation

$$\frac{d^2}{dx^2}\int_{-1}^{1}\chi(\xi)\ln|x-\xi|\,d\xi + \int_{-1}^{1}\chi(\xi)\,R(x,\xi)d\xi = \sqrt{1+k^2}T(x), |x|<1$$

$$R(x,\xi) = (1+k^2)\Omega_{22}(x-\xi,k(x+\xi)+4\xi) + R^*(x-\xi)$$

$$R^*(x-\xi) = \left[(1-k^2)\Omega_{22}\big(x-\xi,k(x-\xi)\big) - 2k\Omega_{12}(x-\xi) - \tfrac{1}{2\pi}(x-\xi)^2\right] \quad (7)$$

$$\Omega_{12}(z) = \frac{1}{4\pi} \frac{\beta^2 sh(\beta z)\sin(\beta z)}{(ch(\beta z) - \cos(\beta kz)^2}, \quad \beta = \frac{\pi}{2l}$$

$$\Omega_{22}(z, y) = -\frac{1}{2\pi}\beta^2 \frac{(ch(\beta z)\cos(\beta y) - 1)}{(ch(\beta z) - \cos(\beta y))^2}$$

It is possible to show that

$$\lim_{z\to 0} R^*(z) = \frac{1}{2}(a + k^2)$$

If to use the expansion of the function $R^*(x,\xi)$ in the series, it is possible to show that the function is continuous one, and moreover, it is infinitely differentiable at point z = 0. Hence, $R(x,\xi)$ is infinitely differentiable too. It allows to use the orthogonal polynomials methods (Popov, 2007) for the equation (7) solving and to find the stress intensity of factor near crack's ends.

5. Conclusions

Thus, the stated problem is reduced to the singular integral equation, allowing the construction of the effective approximate solution (Popov, 1982).

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