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FORCED VIBRATION ANALYSIS OF PRESTRESSED EULER-BERNOULLI BEAM WITH DISCONTINUITIES BY MEANS OF DISTRIBUTIONS WITHOUT USING MODAL ANALYSIS

J. Sobotka*

Abstract: *This paper is a continuation of the previous paper in which the author published the generalized mathematical model for forced vibration of Euler-Bernoulli beam covering discontinuities caused by concentrated loading, concentrated support, concentrated inertia forces or internal hinges. In this new paper, the generalized mathematical model is augmented to cover geometric nonlinearity of stress stiffening or weakening of the beam with the same type of discontinuities. This new analytic approach can offer three advantages. Firstly, steady-state responses of the beam can be found directly without doing modal analysis. Secondly, these responses of the beam are expressed in closed form. Thirdly, remaining continuity conditions at points of the discontinuities are fulfilled automatically. To give an example of using this new approach based on distributions, new closed-form expressions for forced steady-state response of pre-stressed simply supported beam with concentrated harmonic loading are presented.*

Keywords: Vibration, Beam, Discontinuities, Distributions, Dirac, Heaviside.

1. Introduction

Classical analytical method of calculating harmonic steady-state response of the beam is based on the following main steps (Rao, 2007 and Weaver et al., 1990). Firstly, we obtain a frequency equation for specific support conditions of the beam. Secondly, we solve the frequency equation for natural frequencies. Thirdly, we find orthogonal mode shapes corresponding to the natural frequencies of the beam. Finally, we express a forced response of the beam as a linear combination of the mode shapes by finding corresponding modal participation coefficients.

Applying distributional derivative for discontinuous shear force, discontinuous bending moment, and discontinuous rotation of cross section of a beam, we can derive a mathematical model for forced transverse vibration of a beam with discontinuities caused by concentrated supports or concentrated masses or concentrated mass moments of inertia or concentrated transverse forces or concentrated moments situated between ends of the beam, or hinges connecting beam segments. This mathematical model can be solved like only one differential task without dividing the beam into segments where all the continuity conditions among adjoining segments are fulfilled automatically. Using this approach, we have only four integration constants irrespective of the number of the discontinuities. Applying distributions, we do not have to compute natural frequencies, mode shapes or modal participation coefficients in analyzing forced harmonic response of beams.

Various ways of applying distributions can be found in papers by Kozien (2013), Wang (2007), Yesilce (2012), Zhao et al. (2016).

2. The classical equation of motion for forced vibration of pre-stressed Euler-Bernoulli beam

Equation of motion of a beam under distributed transverse force without discontinuities in shear force, in bending moment or in rotation of cross section of the beam is given by (Rao, 2007)

$$\left(\frac{\partial^2}{\partial x^2}\left(E\,\mathrm{J}(x)\left(\frac{\partial^2}{\partial x^2}\mathrm{w}(x,t)\right)\right)\right)-\left(\frac{\partial}{\partial x}\left(N\left(\frac{\partial}{\partial x}\mathrm{w}(x,t)\right)\right)\right)+\rho\,\mathrm{A}(x)\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right)=\mathrm{f}(x,t)\,, \quad (1)$$

* Ing. Jiří Sobotka, PhD., ČEZ, a.s., 67550 Dukovany; CZ, jiri.sobotka@cez.cz

where w(x, t) is transverse displacement of the beam centerline, A(x) is cross-sectional area, J(x) is area moment of inertia, E is modulus of elasticity (Young's modulus), ρ is density, f(x,t) is the distributed transverse force and N is an axil load.

3. A mathematical model for forced transverse vibration of pre-stressed Euler-Bernoulli beam with discontinuities

In order to be able to express possible discontinuities in shear force, bending moment or in rotation of cross section along a centerline of a beam mathematically without cutting the beam into segments which would be without discontinuities, distributional derivative (Schwartz, 1972 and Štěpánek, 2001 and Kanwal, 2004) can be used.

The generalized mathematical model for forced vibration of Euler-Bernoulli beam with various discontinuities without axial loading was published in (Sobotka, 2016) containing distributional derivatives of shear force, bending moment and cross section rotation.

By augmenting the classical part of distributional derivative of shear force in accordance with Eq. (1), new generalized mathematical model for forced vibration of Euler-Bernoulli beam with discontinuities allowing for stress stiffening (in case of tensile axial load) or stress weakening (in case of compressive axial load) can be obtained as follows:

$$\frac{\partial}{\partial x}\mathrm{Q}(x,t) = \rho\,\mathrm{A}(x)\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right) - \left(\frac{\partial}{\partial x}(N\,\phi(x,t))\right) + \left(\sum_{i=1}^{n_1} r_i(t)\,\delta(x-a_i)\right)$$
$$+\left(\sum_{i=1}^{n_2} m_i\left(\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\right)\Bigg|_{x=b_i}\delta(x-b_i)\right) - \left(\sum_{i=1}^{n_3} f_i(t)\,\delta(x-c_i)\right), \qquad (2)$$

$$\frac{\partial}{\partial x}\mathrm{M}(x,t) = \mathrm{Q}(x,t) - \left(\sum_{i=1}^{n_2} J_i\left(\frac{\partial^2}{\partial t^2}\phi(x,t)\right)\Bigg|_{x=b_i}\delta(x-b_i)\right) + \left(\sum_{i=1}^{n_4} s_i(t)\,\delta(x-d_i)\right), \qquad (3)$$

$$\frac{\partial}{\partial x}\phi(x,t) = -\frac{\mathrm{M}(x,t)}{E\,\mathrm{J}(x)} + \left(\sum_{i=1}^{n_5}\psi_i(t)\,\delta(x-e_i)\right), \qquad (4)$$

$$\frac{\partial}{\partial x}\mathrm{w}(x,t) = \phi(x,t)\,. \qquad (5)$$

The right-hand side of Eq. (2) is the distributional derivative of shear force with respect to x where N is axial load (positive if tensile, negative if compressive), $r_i(t)$ is a reaction force at i*th* concentrated support at $x = a_i$ ($0 < a_i < L$), L is total length of the beam, m_i is a concentrated inertia mass at $x = b_i$ ($0 < b_i < L$), $f_i(t)$ is a concentrated transverse load at $x = c_i$ ($0 < c_i < L$), $\delta(x - a_i)$ denotes Dirac's singular distribution moved to a point of the specific discontinuity, n_1 is a number of point supports (without end supports), if any, n_2 is a number of concentrated inertia masses (without end masses), if any, and n_3 is a number of concentrated transverse loads (without end loads), if any.

The right-hand side of Eq. (3) is the distributional derivative of bending moment with respect to x where J_i is a concentrated mass moment of inertia at $x=b_i$, if any, $s_i(t)$ is a concentrated moment load at $x = d_i$ ($0 < d_i < L$), and n_4 is a number of concentrated moment loads (without end moment loads), if any.

The right-hand side of Eq. (4) is the distributional derivative of cross section rotation with respect to x where $\psi_i(t)$ is a magnitude of jump discontinuity in rotation of the cross sections at a hinge connecting segments of the beam at $x = e_i$ ($0 < e_i < L$), and n_5 is a number of internal hinges, if any.

4. Forced vibration solution

Supposing harmonic time variation of loading as

$$f_i(t) = F_i\,\mathbf{e}^{(j\,\omega\,t)}\,, \qquad s_i(t) = S_i\,\mathbf{e}^{(j\,\omega\,t)}\,,$$

and solution to Eqs. (2) to (5) as

$$Q(x,t) = Q_a(x)\,\mathbf{e}^{(j\,\omega\,t)}\ ,\quad M(x,t) = M_a(x)\,\mathbf{e}^{(j\,\omega\,t)}\ ,\quad \phi(x,t) = \phi_a(x)\,\mathbf{e}^{(j\,\omega\,t)}\ ,\quad w(x,t) = w_a(x)\,\mathbf{e}^{(j\,\omega\,t)}$$

$$r_i(t) = R_i\,\mathbf{e}^{(j\,\omega\,t)}\ ,\qquad \psi_i(t) = \Psi_i\,\mathbf{e}^{(j\,\omega\,t)}\qquad ,\ j^2 = -1$$

where ω is circular frequency of vibration, and denoting amplitudes of vibration at points with concentrated inertia masses and moments of inertia as

$$W_i = \lim_{x \to b_i} w_a(x)\ ,\qquad \Phi_i = \lim_{x \to b_i} \phi_a(x)\ , \tag{6}$$

Eqs. (7) to (10) can be derived for unknown general amplitudes of deflection, w_a , rotation of cross section, φ_a , bending moment, M_a , and shear force, Q_a , for a uniform beam as:

$$\frac{d}{dx}Q_a(x) = -\rho\,A\,w_a(x)\,\omega^2 - N\left(\frac{d}{dx}\phi_a(x)\right) + \left(\sum_{i=1}^{n_1} R_i\,\delta(x-a_i)\right) - \left(\sum_{i=1}^{n_2} m_i\,W_i\,\omega^2\,\delta(x-b_i)\right) - \left(\sum_{i=1}^{n_3} F_i\,\delta(x-c_i)\right), \tag{7}$$

$$\frac{d}{dx}M_a(x) = Q_a(x) + \left(\sum_{i=1}^{n_2} J_i\,\Phi_i\,\omega^2\,\delta(x-b_i)\right) + \left(\sum_{i=1}^{n_4} S_i\,\delta(x-d_i)\right), \tag{8}$$

$$\frac{d}{dx}\phi_a(x) = -\frac{M_a(x)}{E\,J} + \left(\sum_{i=1}^{n_5} \Psi_i\,\delta(x-e_i)\right), \tag{9}$$

$$\frac{d}{dx}w_a(x) = \phi_a(x)\,. \tag{10}$$

By using Laplace transform method, general solution to Eqs. (7) to (10) can be computed, in which integration constants are in the form of initial parameters.

As an example of using the proposed method, new closed-form expressions for the amplitude of the deflection, Eq. (11), and for the amplitude of the bending moment, Eq. (12), of forced steady-state response of simply supported (pinned-pinned) uniform beam subjected to a concentrated transverse harmonic force, F.sin(ω.t), at x=a, and to a constant axial force, N, have been obtained as follows:

$$w_a(x) = \left(\frac{\lambda_1^{\,2}\,\lambda_2\,\sin(\lambda_2\,(-x+a))}{(\lambda_1^{\,2}+\lambda_2^{\,2})\,m\,\omega^2} - \frac{\lambda_2^{\,2}\,\sinh(\lambda_1\,(-x+a))\,\lambda_1}{(\lambda_1^{\,2}+\lambda_2^{\,2})\,m\,\omega^2}\right) F\,\mathrm{H}(x-a)$$
$$+\left(-\frac{\lambda_1^{\,2}\,\sin(\lambda_2\,(-l+a))\,\lambda_2\,\sin(\lambda_2\,x)}{\sin(\lambda_2\,l)\,(\lambda_1^{\,2}+\lambda_2^{\,2})\,m\,\omega^2} + \frac{\lambda_2^{\,2}\,\sinh(\lambda_1\,x)\,\sinh(\lambda_1\,(-l+a))\,\lambda_1}{\sinh(\lambda_1\,l)\,(\lambda_1^{\,2}+\lambda_2^{\,2})\,m\,\omega^2}\right) F\,, \tag{11}$$

$$M_a(x) = \left(\frac{\lambda_2\,\sin(\lambda_2\,(-x+a))}{\lambda_1^{\,2}+\lambda_2^{\,2}} + \frac{\lambda_1\,\sinh(\lambda_1\,(-x+a))}{\lambda_1^{\,2}+\lambda_2^{\,2}}\right) F\,\mathrm{H}(x-a)$$
$$+\left(-\frac{\lambda_2\,\sin(\lambda_2\,(-l+a))\,\sin(\lambda_2\,x)}{(\lambda_1^{\,2}+\lambda_2^{\,2})\,\sin(\lambda_2\,l)} - \frac{\lambda_1\,\sinh(\lambda_1\,x)\,\sinh(\lambda_1\,(-l+a))}{(\lambda_1^{\,2}+\lambda_2^{\,2})\,\sinh(\lambda_1\,l)}\right) F\ , \tag{12}$$

where

$$m = \rho\,A\ ,\qquad \omega = 2\,\pi\,f\ , \tag{13}$$

$$\lambda_1 = \frac{\sqrt{2}\,\sqrt{\dfrac{N+\sqrt{N^2+4\,E\,J\,m\,\omega^2}}{E\,J}}}{2}\ ,\qquad \lambda_2 = \frac{\sqrt{2}\,\sqrt{\dfrac{-N+\sqrt{N^2+4\,E\,J\,m\,\omega^2}}{E\,J}}}{2}\ . \tag{14}$$

Notation H(x-a) in Eq. (11), (12) is used for Heaviside’s unit step function. When finding a value of the defection at the point of the concentrated load, it is important for a limit of the right hand side of Eq. (11) to be evaluated as follows:

$$\lim_{x \to a} w_a(x) = \frac{1}{2} \frac{\lambda_2^{\,2} \lambda_1 \sinh(-\lambda_1 l + \lambda_1 a) F \mathbf{e}^{(\lambda_1 a)}}{\sinh(\lambda_1 l)(\lambda_1^{\,2} + \lambda_2^{\,2}) \omega^2 m} + \frac{1}{2} \frac{\lambda_2 \lambda_1^{\,2} (-\cos(\lambda_2 l) + \cos(2 \lambda_2 a - \lambda_2 l)) F}{\sin(\lambda_2 l)(\lambda_1^{\,2} + \lambda_2^{\,2}) m \omega^2} - \frac{1}{2} \frac{\lambda_2^{\,2} \lambda_1 \sinh(-\lambda_1 l + \lambda_1 a) F \mathbf{e}^{(-\lambda_1 a)}}{\sinh(\lambda_1 l)(\lambda_1^{\,2} + \lambda_2^{\,2}) \omega^2 m} . \tag{15}$$

Correctness of Eq. (11), (12), (15) has been verified by using FEM.

5. Conclusions

Equations (2) to (5) making the generalized mathematical model for forced response of pre-stressed Euler-Bernoulli beam with different discontinuities are the first contribution of this paper to vibration analysis of beams.

In this model, each unknown dependently variable quantity has got its own distributional derivative covering specific jump discontinuities where Dirac‘s singular distribution, denoted here as $\delta(x)$, is always moved to the point with a specific jump discontinuity, and multiplied by a magnitude of the discontinuity. Discontinuities in shear force are supposed to be owing to idealized concentrated supports or inertia masses or concentrated transverse forces situated between ends of the beam. Likewise, discontinuities in bending moment are assumed to be due to idealized concentrated moments of inertia or concentrated moment loads situated between ends of the beam. Discontinuities in cross section rotation may be caused by real hinges connecting beam segments.

To be able to find forced steady-state response of Euler-Bernoulli beams analytically with discontinuities mentioned allowing for stress stiffening or weakening of the beam, equations (7) to (10) have been derived as the second contribution of this paper for unknown amplitudes of shear force, bending moment, cross section rotation and deflection of the beam. When using these equations, boundary conditions are given directly, which means that internal forces are not expressed through the deflection and its derivatives.

By using this proposed approach, new equations (11), (12), (15) for forced steady-state response of pre-stressed simply supported beam subjected to concentrated transverse harmonic loading have been obtained as the third contribution. These new expressions have got the closed form, which would not be possible if modal analysis were used. The right hand side of Eq. (11), (12), (15) is exact in the sense of Euler-Bernoulli theory. The effect of stress stiffening in Eq. (11), (12), (15) is covered by choosing a positive value for axial force. The effect of stress weakening in Eq. (11), (12), (15) is covered by choosing a negative value for axial force not allowing buckling of the beam.

References

Kanwal, R.P. (2004) Generalized Functions. Birkhäuser, Boston.

Kozien, M.S. (2013) Analytical Solutions of Excited Vibrations of a Beam with Application of Distribution. ACTA PHYSICA POLONICA A, Vol. 123, No. 6, pp. 1029-1033.

Rao, S.S. (2007) Vibration of Continuous Systems. Wiley, New Jersey.

Schwartz, L. and Brabec, J. (1972) Mathematical methods in physics. SNTL, Prague (in Czech).

Sobotka, J. (2016) Forced vibration analysis of Euler-Bernoulli beam with discontinuities by means of distributions without doing modal analysis, in: Proc. 22nd Int. Conf. on Engineering Mechanics (eds. Zolotarev, I. and Radolf, V.), Svratka, Czech Republic, pp. 510-513.

Štěpánek, J. (2001) Distributions and differential equations. Karolinum, Prague (in Czech).

Wang, J. and Qiao, P. (2007) Vibration of beams with arbitrary discontinuities and boundary conditions. Journal of Sound and Vibration, Vol. 308, Is. 1, pp. 12-27.

Weaver, W., Timoshenko, S.P. and Young, D.H. (1990) Vibration Problems in Engineering, 5th ed., John Wiley & Sons, Inc.

Yesilce, Y. (2012) Free and forced vibrations of an axially-loaded Timoshenko multi-span beam carrying a number of various concentrated elements. Shock and Vibration, Vol. 19, No. 4, pp. 735-752.

Zhao, X., Zhao, Y. R., Gao, X. Z., Li, X. Y. and Li, Y. H. (2016) Green's functions for the forced vibrations of cracked Euler–Bernoulli beams. Mechanical Systems and Signal Processing, Vol. 68, pp. 155-175.