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CONSTRUCTION OF RANDOM FIELD BASED ON IMAGE ANALYSIS

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Abstract: *The principal challenge in implementation of random fields arises from the need for determination of their correlation/characteristic lengths in the simplest case or more generally their covariance functions. The present contribution is devoted to the construction of random fields based on image analysis utilising statistical descriptors, which were developed to describe the different morphology of random material. A numerical study of one-dimensional images is performed in order to investigate the quality of obtained random fields.*

Keywords: Random fields, Two-point probability function, Covariance function, Karhunen-Loève expansion, Principal component analysis.

1. Introduction

When dealing with a heterogeneous material, some material parameters can vary spatially in an uncertain fashion and therefore random fields are suitable for their description. In a computational setting, the random field and the numerical model must be discretized. Therefore, the most common approach for achieving this is the Karhunen-Loève expansion (KLE), see (Kučerová et al., 2012). The KLE allows for representation of random fields utilising surprisingly few orthogonal terms from spectral decomposition of covariance function, see (Adler and Taylor, 2007). Several analytical covariance functions were developed to describe the spatial variability, but their relevance in describing real material properties remains questionable and poorly identified. Recently, relatively new techniques of extracting the spatial randomness from images were developed, see (Soize, 2006; Jürgens et al., 2012).

Here, we introduce a novel construction of covariance function based on the two-point probability density function (see Torquato, 2002), which is calculated from the given discretised image. Due to the limited space it is impossible to present the mathematical formulation of entire methodology. Therefore, we refer only basic features of each procedure used for the different constructions of random fields.

- Karhunen-Loève expansion. It is an extremely useful tool for the concise representation of the stochastic processes. Based on the spectral decomposition of covariance function, the KLE decomposes the process into a series of orthogonal functions with the random coefficients, see (Adler and Taylor, 2007). For practical implementation, the KLE is truncated after M terms, yielding the suitable approximation. Based on the spectral decomposition of covariance function $C(\boldsymbol{x}, \boldsymbol{x}')$ and the orthogonality of eigenfunctions ϕ_i, the real-valued random field $\lambda(\boldsymbol{x}, \omega)$ can be written as

$$\lambda(\boldsymbol{x}, \omega) \approx \mu_\lambda(\boldsymbol{x}) + \sum_{i=1}^{M} \sqrt{\zeta_i}\xi_i(\omega)\phi_i(\boldsymbol{x}), \tag{1}$$

  where $\mu_\lambda(\boldsymbol{x})$ is the mean value, ζ_i are the positive eigenvalues and $\boldsymbol{\xi}(\omega)$ is a set of uncorrelated random variables of zero mean and unit variance.
- Two-point probability function (S_2). It is a statistical descriptor developed for the morphology description of multi-phase random heterogeneous material, see (Havelka et al., 2016). Here, the

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two-point probability function is utilised for the computation of covariance function.

– Covariance function (CF). It is a spatial measure of how much two variables change together and plays a key role in the construction of random field. There are several well-known analytical functions (e.g. Exponential, Gaussian, see (Kučerová and Sýkora, 2013)), and/or it can be directly calculated from the images employing the statistical descriptor, see (Lombardo et al., 2009) for transformation formula.
– Principal component analysis (PCA). It is an orthogonal linear transformation that transforms a set of correlated variables into a set of linearly uncorrelated variables called principal components, see (Jolliffe, 2002).

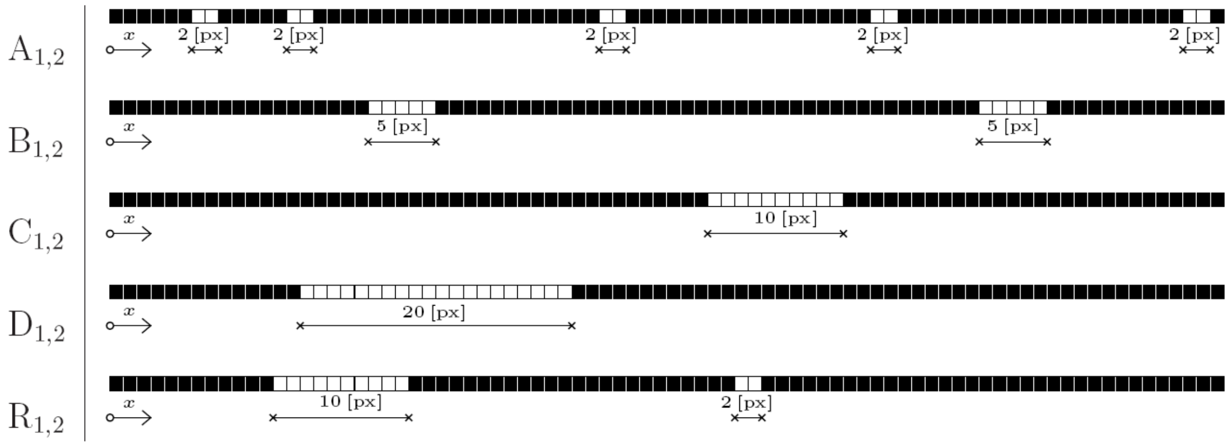

Fig. 1: Reference media, size 1 x 100000 px: A_1 – particles 1 x 2 px filling 10 % of volume; A_2 – particles 1 x 2 px filling 50 % of volume; B_1 – particles 1 x 5 px filling 10 % of volume; B_2 – particles 1 x 5 px filling 50 % of volume; C_1 – particles 1 x 10 px filling 10 % of volume; C_2 – particles 1 x 10 px filling 50 % of volume; D_1 – particles 1 x 20 px filling 10 % of volume; D_2 – particles 1 x 20 px filling 50 % of volume; R_1 – particles ranging from 1 x 2 px to 1 x 20 px filling 10 % of volume; R_2 – particles ranging from 1 x 2 px to 1 x 20 px filling 50 % of volume.

2. Numerical examples

The first part of numerical analysis is devoted to the identification of the correlation lengths used in the analytical relations of covariance functions. As an illustration, we utilised a set of digitised images representing artificially created particulate suspensions consisting of rectangular white particles randomly distributed within a black matrix. The initial binary structures 1 x 100000 px and their basic statistical properties are shown in Fig. 1 and in Tab. 1, respectively.

Tab. 1: Basic statistical properties.

		A_1	A_2	B_1	B_2	C_1	C_2	D_1	D_2	R_1	R_2
Vol. '0'	[%]	90.0	50.0	90.0	50.0	90.0	50.0	90.0	50.0	90.0	50.0
μ	[-]	0.10	0.50	0.10	0.50	0.10	0.50	0.10	0.50	0.10	0.50
σ	[-]	0.30	0.50	0.30	0.50	0.30	0.50	0.30	0.50	0.30	0.50

The calibration procedure of the covariance lengths is relatively simple and intuitive process. The proper optimization algorithm is used to minimise the difference between the original covariance function calculated from two-point probability function of reference medium (see (Lombardo et al., 2009)) and the computed one. In our study, we utilised the in-house GRADE algorithm, which is a real-coded stochastic optimization algorithm combining the principles of genetic algorithms and differential evolution, see (Ibrahimbegović et al., 2004). The results obtained for two covariance kernels, i.e. Gaussian and Exponential, are summarised in Tab. 2.

In the next example, several numerical constructions of random fields from input digitised images are examined to achieve a real description of spatial variability. To keep this study clear, let us consider the same set of one-dimensional binary images as in the previous example, see Fig. 1. The results in Tab. 2 show that the computed correlation lengths are approximately ten thousand times smaller than the original dimensions of the investigated structure. Thanks to this fact we can reduce the dimensions of our problem to 1 x 100 px, and thus decrease the computational demands to a reasonable level. Besides that, it is necessary to objectively assess the construction techniques of random field in terms of their accuracy. For this purpose we prepare a verification set for each reference medium consisting of 10000 images with dimensions 1 x 100 px randomly cut from the original image.

Tab. 2: Optimised covariance lengths calibrated for Gaussian (GK) and Exponential kernel (EK).

		A_1	A_2	B_1	B_2	C_1	C_2	D_1	D_2	R_1	R_2
GK - l_x	[px]	0.75	0.50	1.91	1.10	3.78	2.33	7.66	4.77	4.91	2.48
EK - l_x	[px]	0.87	0.50	2.39	1.20	4.68	2.59	9.64	5.40	6.37	3.10

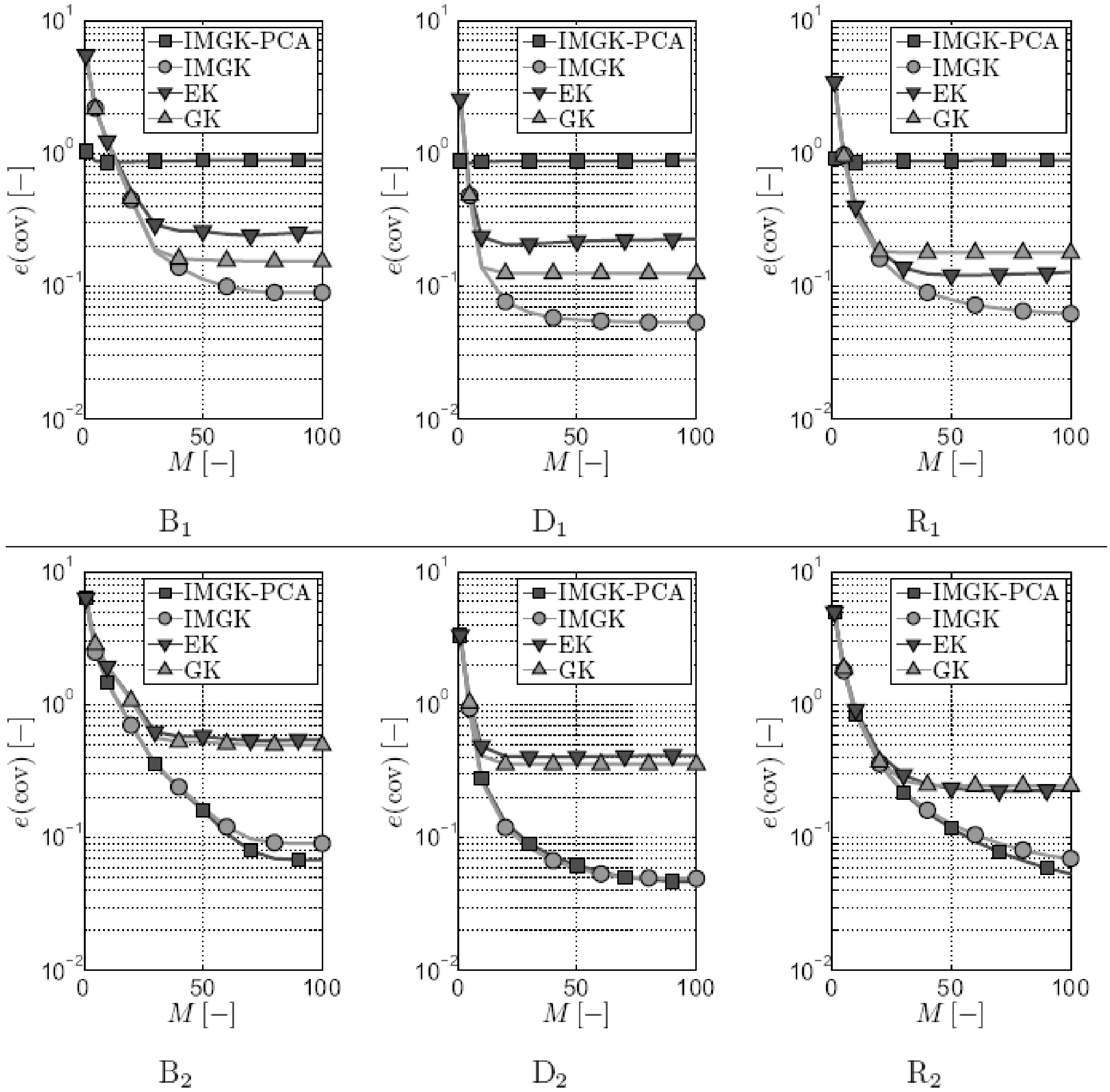

Fig. 2: Relative error of covariance matrix as a function of KLE terms M [-] calculated for reference media - B_1, B_2, D_1, D_2, R_1, R_2.

Overall, four methods for a construction of random fields: (i) Gaussian-based (GK), (ii) Exponential-based (EK), (iii) Image-based (IMGK), and (iv) Image-PCA-based (IMGK-PCA) were examined for 10000 realisations. The errors *e*(cov) on the prediction of covariance computed relatively to the verification sets are plotted as a function of the number of KLE terms in Fig. 2. It can be seen the gain of image-based random fields' construction and strong dependencies of relative errors for a small number of KLE modes.

3. Conclusions

In this contribution, we present different strategies for construction of random fields. A comparison of classical approach based on the analytical covariance functions, namely Exponential and Gaussian, and a novel methodology based on image analysis was shown to assess the quality and accuracy of obtained random fields. The whole concept was demonstrated on a digitised binary image of two phase medium.

The most interesting finding is that the image-based random fields' construction provides more precise description of spatial variability than constructions based on the analytical covariance functions with optimised correlation lengths. It is a probably logical conclusion, but the use of analytical covariance functions, especially without calibrated correlation lengths, in random fields' construction is very widespread technique in numerical modelling of heterogeneous material and leads evidently to inaccurate results.

Another important result is related to the truncation of the Karhunen- Loève expansion. It is evident from the presented figures that the proposed methodology is very sensitive to small number of Karhunen-Loève terms. In this region, the relative error of random fields compared to verification set decreases very sharply.

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