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FREE VIBRATION ANALYSIS OF LAMINATED GLASS BEAMS USING DYNAMIC EFFECTIVE THICKNESS AND OTHER APPROACHES

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Abstract: *Effective thickness approaches are useful tools for a response prediction and the design of sandwich structures. In this contribution, we study their applicability to free vibration analysis of laminated glass beams – sandwich structures composed of glass layers connected with one or multiple compliant foils. These interlayers are made of polymer materials with frequency/temperature-dependent behavior. Here, the dynamic effective thickness approach, the modal strain energy method, and the Newton-type algorithm are applied to the complex eigenvalue problem for a three-layered laminated glass beam. The results of the modal analyses, in terms of natural frequencies and loss factors, are compared for all approaches. It is shown that the errors of the two simplified methods depend on the ambient temperature and the applied boundary conditions and that these errors can be large, especially for the loss factor.*

Keywords: Laminated glass, Free vibration, Dynamic effective thickness, Modal strain energy method.

1. Introduction

Laminated glass structures are thin sandwich plates or beams composed of a few glass layers connected with one or more compliant foils, see Fig. 1. These interlayers are made of polymer materials with frequency-dependent and temperature-sensitive behavior. In this contribution, we focus on the modal analysis of such multi-layered structures composed of elastic and viscoelastic layers. A few approaches to the free vibration problem of a viscoelastically damped sandwich can be found in literature. Three of them, two numerical approaches and one semi-analytical method, are applied in this paper to the analysis of laminated glass beams.

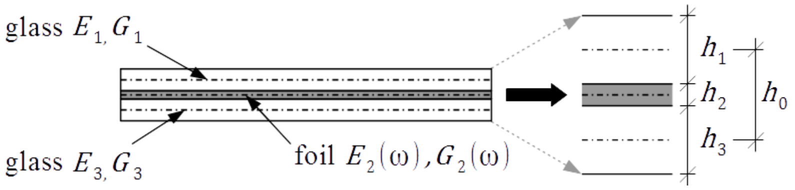

Fig. 1: Laminated glass sandwich's configuration.

2. Semi-analytical and numerical methods for free vibration analysis

The problem is introduced for a laminated glass beam with three layers (glass/interlayer/glass). The viscoelastic behavior of a polymer foil is described by a generalized Maxwell chain model and its damping behavior is accounted for through the complex shear modulus of the foil

$$G_2(\omega) = G_0 - \sum_{p=1}^{P} \left(G_p \frac{1}{\omega^2 \theta_p^2 + 1} \right) + \mathrm{i} \sum_{p=1}^{P} \left(G_p \frac{\omega}{\omega^2 \theta_p^2 + 1} \right) = G_0 + G_\omega(\omega), \qquad (1)$$

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where ω is the complex value of an angular frequency (or its real part), G_0 is the elastic shear modulus of the whole chain, P stands for the number of viscoelastic units, G_p denotes the shear modulus of the p-th unit, θ_p is its relaxation time, and $G_\omega(\omega)$ is the frequency-dependent part of the shear modulus. Therefore, the eigenvalue problem corresponding to the free vibration of a laminated glass beam is nonlinear and the eigenvalues and eigenvectors are complex. Three methods for this task are discussed in this paper: the dynamic effective thickness approach, the Newton-type algorithm for the complex eigenvalue problem, and the modal strain energy method. All methods were implemented into our MATLAB-based solver.

2.1. Dynamic effective thickness approach

To the best of our knowledge, a few effective thickness formulations can be found in literature for laminated glass beams and plates under static loading, whereas only one effective thickness approach exists for dynamic problems (López-Aenlle et al., 2014). The effective thickness is derived from the analytical model by Ross et al. (1959). The authors considered a three-layered, simply-supported beam with purely elastic face layers and a linearly viscoelastic core with a complex shear modulus. The shear strains in face layers are neglected, whereas only shear stresses are assumed in the core. The deflection is the same for all layers, and there is no slipping at layer-interfaces. Under these assumptions, the fourth-order differential equation of bending wave motion with the effective complex bending stiffness can be expressed. Then, the dynamic effective thickness for laminated glass beams, derived from the effective complex bending stiffness formulated in (Ross et al., 1959), is given by

$$h_{\text{eff}}(\widetilde{\omega}) = \sqrt[3]{(h_1^3 + h_3^3)\left(1 + Y\left(1 + \frac{h_1}{g(\widetilde{\omega})(h_1+h_3)}\right)^{-1}\right)}. \qquad (2)$$

In this approach, the effective thickness is a function of a real angular frequency $\widetilde{\omega}$. The geometric parameter is provided by

$$Y = \frac{12h_0h_1h_3}{(h_1^3+h_3^3)(h_1+h_3)} \qquad (3)$$

with the thicknesses h_1 and h_3 and the distance h_0 introduced in Fig. 1. The shear parameter reads

$$g(\widetilde{\omega}) = \frac{G_2(\widetilde{\omega})}{E_3h_3h_2k^2} \qquad (4)$$

and combines the complex shear modulus of the interlayer $G_2(\omega)$, the Young modulus of glass $E_3 = E_1$, the thicknesses of layers h_2 and h_3, and the wavenumber k.

Due to the frequency-dependency, the problem is solved by an iterative algorithm. The initial frequency $\widetilde{\omega}_k = \widetilde{\omega}_0$ is set to the average of the frequencies of two limiting cases: a monolithic glass beam with perfect interaction of glass layers and two independent monolithic glass layers with no interaction. The natural frequency f and the modal loss factor η follow from the equations (López-Aenlle et al., 2014)

$$\widetilde{\omega}_{k+1}^2(1 + \mathrm{i}\eta) = k^4\frac{E_3h_{\text{eff}}^3(\widetilde{\omega}_k)}{12\bar{m}}, \quad f = \frac{\widetilde{\omega}}{2\pi}, \qquad (5)$$

where $\bar{m}$ is the mass per unit length. If the error of the new and the previous values of the frequency and the loss factor is out of the tolerance limit, the dynamic effective thickness is updated for the new frequency and the new values of the natural frequency and the loss factor are computed.

2.2. Complex-frequency approach using finite element method and Newton-type algorithm

Using the finite element discretization, the mass matrix $\boldsymbol{M}$ of the laminated glass sandwich is real-valued and frequency-independent, whereas the stiffness matrix $\boldsymbol{K}(\omega)$ is complex,

$$\boldsymbol{K}(\omega) = \boldsymbol{K}_0 + G_\omega(\omega)\boldsymbol{K}_{\text{const}}, \qquad (6)$$

and consists of the frequency-independent part $\boldsymbol{K}_0$ (corresponding to the glass layers and to the elastic part of the shear modulus of the foil G_0) and the frequency-dependent part $G_\omega(\omega)\boldsymbol{K}_{\text{const}}$ (corresponding to the frequency-dependent part of the foil shear modulus $G_\omega(\omega)$; $\boldsymbol{K}_{\text{const}}$ is a constant matrix), see for example (Daya et al., 2001).

The well-known natural vibration problem can be written in the form

$$(\boldsymbol{K}(\omega) - \omega^2\boldsymbol{M})\boldsymbol{U} = \boldsymbol{0}, \qquad (7)$$

where ω stands for a complex-valued angular frequency and the mode shape $\boldsymbol{U}$ is a complex nodal vibration eigenvector. To be well-posed, the problem is complemented with the additional equation

$$\boldsymbol{U}_0^{\mathrm{T}}(\boldsymbol{U} - \boldsymbol{U}_0) = 0, \tag{8}$$

where $\boldsymbol{U}_0$ is the solution of the real-eigenvalue problem

$$(\boldsymbol{K}_0 - {\omega_0}^2 \boldsymbol{M})\boldsymbol{U}_0 = \boldsymbol{0} \tag{9}$$

obtained by a built-in MATLAB solver.

The system of nonlinear equations (7) and (8) can be solved by the Newton method starting from the real-eigenvalue solution of equation (9), $\boldsymbol{U}_k = \boldsymbol{U}_0$ and $\omega_k = \omega_0$. Next, the mode shapes and the frequencies are updated

$$\boldsymbol{U}_{k+1} = \boldsymbol{U}_k + \delta\boldsymbol{U}_{k+1}, \quad \omega_{k+1} = \omega_k + \delta\omega_{k+1} \tag{10}$$

with increments $\delta\boldsymbol{U}_{k+1}$ and $\delta\omega_k$. These increments result from the system of equations

$$\begin{bmatrix} \boldsymbol{K}(\omega_k) - \omega_k^2\boldsymbol{M} & \left(\frac{\partial \boldsymbol{K}}{\partial \omega}(\omega_k) - 2\omega_k\boldsymbol{M}\right)\boldsymbol{U}_k \\ \boldsymbol{U}_0^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{U}_{k+1} \\ \delta\omega_{k+1} \end{bmatrix} = -\begin{bmatrix} \left(\boldsymbol{K}(\omega_k) - \omega_k^2\boldsymbol{M}\right)\boldsymbol{U}_k \\ \boldsymbol{U}_0^{\mathrm{T}}(\boldsymbol{U}_k - \boldsymbol{U}_0) \end{bmatrix}. \tag{11}$$

We repeat this update until the norm of the right-hand side does not exceed a given tolerance. The eigenvalues and the eigenvectors are complex. The natural frequency f and the loss factor η are calculated from the complex value ω according to

$$\omega^2 = \widetilde{\omega}^2(1 + \mathrm{i}\eta), \quad f = \frac{\widetilde{\omega}}{2\pi}. \tag{12}$$

2.3. Real-frequency approach using finite element method and modal strain energy method

The complex eigenvalue solution can be expensive for larger problems. Therefore, the modal strain energy method was introduced by Johnson et al. (1982) to overcome this difficulty. The approximated value of modal loss factor of sandwich beams

$$\eta \approx \eta_m \, \Pi_2/\Pi \tag{13}$$

involves the material loss factor of the core material (foil)

$$\eta_m = \mathrm{Im}[G_2(\omega_k)]/\mathrm{Re}[G_2(\omega_k)], \tag{14}$$

the elastic strain energy associated with the given mode shape Π, and the elastic strain energy attributed to the viscoelastic core Π_2, which are given by

$$\Pi = \frac{1}{2}\boldsymbol{U}_{\mathrm{r}}^{\mathrm{T}}\boldsymbol{K}_{\mathrm{r}}(\omega_k)\boldsymbol{U}_{\mathrm{r}}, \quad \Pi_2 = \frac{1}{2}\boldsymbol{U}_{\mathrm{r}}^{\mathrm{T}}\boldsymbol{K}_{\mathrm{r},2}(\omega_k)\boldsymbol{U}_{\mathrm{r}}. \tag{15}$$

Here, $\boldsymbol{U}_{\mathrm{r}} = \mathrm{Re}[\boldsymbol{U}]$ stands for the corresponding undamped mode shape and the matrix $\boldsymbol{K}_{\mathrm{r}} = \mathrm{Re}[\boldsymbol{K}(\omega_k)]$ consists of the stiffness matrices of the glass layers and the real part of the stiffness matrix of the foil $\boldsymbol{K}_{\mathrm{r},2} = \mathrm{Re}[G_2(\omega)]\boldsymbol{K}_{\mathrm{const}}$. The natural frequencies and the mode shapes are determined iteratively by consecutive solutions of the undamped real-valued eigenproblem

$$\left(\boldsymbol{K}_{\mathrm{r}}(\omega_k) - \omega_{k+1}^2\boldsymbol{M}\right)\boldsymbol{U}_{\mathrm{r},k+1} = \boldsymbol{0}, \tag{16}$$

and the approximated loss factors are computed from (13).

3. Results

The results of modal analyses for the previous three approaches are compared for a simply-supported and a clamped-clamped laminated glass beam at ambient temperatures 20 °C and 40 °C. The dimensions of the beam are: the length 1 m, the width 0.1 m, and the thicknesses $h_1/h_2/h_3$ = 10/0.76/10 mm of the glass/foil/glass layers. The material parameters are taken from (López-Aenlle et al., 2014).

In Tab. 1, the modal responses are compared in terms of natural frequencies and loss factors for both types of the boundary conditions and for both temperatures. The results for finite element approaches are computed for 100 elements per the beam length. The tolerance limit, set to 10^{-5}, is used for all methods.

The dynamic effective thickness (DET) approach gives the natural frequencies with the errors less than 1 % compared with those from the Newton method (NM) for the simply-supported beam at both temperatures and also for the clamped-clamped beam at room temperature; the errors are less than 1 %.

However, the errors in natural frequencies are approximately 10 % for the clamped-clamped beam at 40 °C. The modal loss factors from the DET match the NM result only for simply-supported beam at 20 °C. In the other cases, the errors in loss factors are higher: 5 – 10 % for the simply-supported beam at 40 °C and 15 – 50 % for the clamped-clamped beam.

The modal strain energy (MSE) method gives the natural frequencies with the errors less than 1 % compared with those from the NM for both boundary conditions at 20 °C and 5 – 8 % for both beams at 40 °C. The results for the modal loss factor does not match those from NM. The errors are 5 – 15 % for the clamped-clamped beam and 10 – 50 % for the simply-supported beam.

Tab. 1: Comparison of natural frequencies and loss factors provided by the dynamic effective thickness (DET) approach, the modal strain energy (MSE) approximation, and the Newton method (NM).

		simply-supported beam			**clamped-clamped beam**		
temperature 20 °C	Mode	DET	MSE	NM	DET	MSE	NM
natural frequency	1	50.46	50.26	50.42	113.3	113.5	114.2
[Hz]	2	197.4	196.1	196.8	303.8	301.9	303.6
modal loss factor	1	1.22	1.53	1.22	1.74	3.78	3.31
[%]	2	2.22	2.43	2.21	2.88	4.51	4.23
temperature 40 °C							
natural frequency	1	44.66	41.76	44.77	95.16	80.80	85.43
[Hz]	2	160.3	148.0	160.7	241.4	214.2	226.6
modal loss factor	1	18.89	29.84	19.97	24.79	37.28	33.80
[%]	2	28.57	39.85	32.08	29.52	37.04	35.20

4. Conclusions

For the natural vibration problem of a laminated glass beam with a viscoelastic interlayer foil, the results of two simplified methods were compared with the complex eigenvalue solution provided by the Newton-type algorithm. We made a comparison of natural frequencies and loss factors for a simply-supported and a clamped-clamped laminated glass beam at 20 °C and 40 °C.

The dynamic effective thickness approach combines an easy iterative algorithm with simple analytical equations for natural frequencies and loss factors. For frequencies, it gives very good results for simply-supported beam at both temperatures and for the clamped-clamped beam at 20 °C. For loss factors, the errors are under 1 % only for the simply-supported beam at 20 °C.

The modal strain energy method reduces the computational cost of the problem because it works only with the real parts of the mode shapes, the frequency, and the stiffness matrix. It provides very good results for frequencies of both beams at 20 °C. However, the errors in loss factors range from 5 % up to 50 % for the considered examples.

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