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A DAMPED HARMONIC OSCILLATOR IN THE CLASSICAL AND FRACTIONAL DIFFERENTIAL CALCULUS WITH THE LIOUVILLE DERIVATIVE

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Abstract: *This paper considers a fractional differential equation with a Liouville fractional derivative for damped harmonic oscillator. The proposed analytical solution for the fractional equation is compared with the solution for the classical equation. The study involved determining the conditions of the agreement of the two solutions and proposing the physical interpretation of the fractional derivative.*

Keywords: damped harmonic oscillator, fractional differential calculus, Liouville derivative

1. Introduction

In the recent years, researchers have paid a lot of attention to a new mathematical method – the fractional differential calculus. It is expected that the fractional calculus will enable new discoveries and offer a new perspective on the old well-known problems. This study offers a contribution to this very promising area of research.

The harmonic oscillator is fundamental to many theories and models in physics and mechanics. Generally, the equations of the harmonic oscillator and their solutions in the classical calculus are very well known. Some scientists attempt to look at them in a different way by replacing the classical (total) derivatives with fractional ones. In this way, a set of new fractional equations of the harmonic oscillator is being created. The new equations and their solutions are extensively discussed in the literature (e.g. Atanackovic 2014, Blasiak 2017, Herrmann 2014, Kilbas 2006, Podlubny 1999, Stanislavsky 2005).

This paper discusses the fractional equation for a damped harmonic oscillator in the form proposed by R. Herrmann (Herrmann 2014), using the Liouville definition of the fractional derivative (Kilbas 2006, Samko 1993, Podlubny 1999, Uchaikin 2013).

2. General information and definitions

The classical damped harmonic oscillator is described by:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{1}$$

where $x = x(t)$ – the displacement which is a function of time, *m (kq)* – the mass, *c (Ns/m)* – the damping factor, *k (N/m)* – the rigidity factor. Equation (1) has the following analytical solution (*c =”classical”*):

$$x_c(t) = C_1 e^{\omega_{c1} t} + C_2 e^{\omega_{c2} t} \tag{2}$$

$$\omega_{c1} = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad , \quad \omega_{c2} = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{3}$$

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where C_1, C_2 are constants derived from the initial conditions $x(0) = x_0$, $v(0) = v_0$.

Alternatively, the damped harmonic oscillator can be described by means of the fractional equation proposed by R. Herrmann (Herrmann, 2014):

$$m\ddot{x}(t) + \mu x^{(\alpha)}(t) = 0 \tag{4}$$

where $x^{(\alpha)}(t)$ is the fractional derivative of an order of α, $(0 < Re(\alpha) < 1)$.

This paper considers equation (4) with the Liouville fractional derivative (Kilbas, 2006, Podlubny, 1999, Samko, 1993, Uchaikin, 2013):

$$x^{(\alpha)}(t) = {}_L D_+^{\propto} x(t) = \frac{1}{\Gamma(1-\propto)} \frac{d}{dt} \int_{-\infty}^{t} \frac{x(\xi)d\xi}{(t-\xi)^{\alpha-[Re(\propto)]}} \;,\; 0 < Re(\propto) < 1, t\epsilon\mathbb{R} \tag{5}$$

where $x^{(\alpha)}(t)$ is fractional derivative of an order of α ; with α being a complex number.

Then equation (4) can be solved with the ansatz $x(t) = e^{\omega t}$ and with a basic property of the Liouville derivative (Kilbas, 2006):

$$(e^{\omega t})^{(\alpha)} = {}_L D_+^{\propto}(e^{\omega t}) = \omega^{\alpha} e^{\omega t} \;,\; 0 \leq Re(\propto) \tag{6}$$

The characteristic equation of eq. (4) $m\omega^2+\mu\omega^\alpha=0$

$$\omega^{\alpha}\left[\omega^{2-\alpha} + \frac{\mu}{m}\right] = 0, \quad 0 < Re(\propto) < 1 \tag{7}$$

gives:

$$\omega_{f0} = 0 \;,\; \omega_{f1} = exp\left(\frac{ln\left(-\frac{\mu}{m}\right)}{2-\alpha}\right) \;,\; \omega_{f2} = \overline{\omega_{f1}} \tag{8}$$

where f = *"fractional"*. So, for α: $0 < Re(\alpha) < 1$, eq. (4) has an analytical solution:

$$x_f(t) = A_0 e^{\omega_{f0} t} + A_1 e^{\omega_{f1} t} + A_2 e^{\omega_{f2} t} = A_0 + A_1 e^{\omega_{f1} t} + A_2 e^{\omega_{f2} t} \tag{9}$$

where A_0, A_1, A_2 are constants which should be determined using initial conditions. The problem of the three different constants and of the reasonable initial conditions is discussed in (Herrmann, 2014).

3. Problem

All the coefficients in the classical equation (1) are well defined, have a physical sense and are known (experimentally assigned for a particular system). However, in the fractional equation (4), the order of the fractional derivative and the coefficient remains unrecognized, particularly in the context of the physical sense. Thus naturally, the fundamental question appears: what should the order of the fractional derivative α and the coefficient μ be?

In this work, we ask what the order of the fractional derivative α and the coefficient μ should be to make the above solutions of the fractional and classical equations equal ? Obviously, the following relationships should be satisfied:

$$\text{for } \alpha \to 0 \,,\; x^{(\alpha)}(t) \to x^{(0)}(t) = x(t) : \; \mu \to k \,,\; eq. (4) \to m\ddot{x} + kx = 0 \tag{10}$$

$$\text{for } \alpha \to 1 \,,\; x^{(\alpha)}(t) \to x^{(1)}(t) = \dot{x}(t) : \; \mu \to c \,,\; eq. (4) \to m\ddot{x} + c\dot{x} = 0 \tag{11}$$

4. Method

In the study, we compared the solutions of the classical and fractional equations:

$$x_c(t) = x_f(t) \tag{12}$$

$$0 = A_0 \,, C_1 = A_1 \,, C_2 = A_2 \,, \omega_{c1} = \omega_{f1} \,, \omega_{c2} = \omega_{f2}, \omega_{c1} = \omega_{f2} \,, \omega_{c2} = \omega_{f1} \tag{13}$$

Four possibilities appeared in the domain of frequency. We performed calculations for all of the cases and we selected the one for which the solutions (classical and fractional) equal and $0 < Re(\alpha) < 1$ (equations (10) and (11)). Then, this case was used to calculate α and μ. Thus:

$$\omega_{c1} = \omega_{f1} \;\Rightarrow\; -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = exp\left(\frac{ln\left(-\frac{\mu}{m}\right)}{2-\alpha}\right) \tag{14}$$

From equation (14) α is:

$$\alpha = 2 - \frac{ln\left(-\frac{\mu}{m}\right)}{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}} \tag{15}$$

Putting in eq. (15) $\alpha = 0$ and $\alpha = 1$ respectively, we obtained following formulas for μ:

$$for\ \ \alpha = 1 \qquad \mu = -m\left(-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right) \tag{16}$$

$$for\ \ \alpha = 0 \qquad \mu = -m\left(-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right)^2 \tag{17}$$

On the basis of the above and of conditions (10), (11), we postulate the following form of the parameter μ:

$$\mu = -m\left(-\frac{\alpha c}{2m} - \sqrt{\left(\frac{\alpha c}{2m}\right)^2 - \frac{(1-\alpha)k}{m}}\right)^{2-\alpha} \tag{18}$$

And now by returning to equation (15), we obtained the final formula for the order of the fractional derivative α :

$$\alpha = \alpha(m, c, k) = 2 - \frac{c\left(\sqrt{c^2 - 4km} + c\right)}{c\sqrt{c^2 - 4km} + c^2 + 2km} \tag{19}$$

Thus, $\boldsymbol{\alpha = \alpha(m,c,k)}$ and $\boldsymbol{\mu=\mu(\alpha,m,c,k)}$, and satisfy the following conditions:

$$\lim_{c\to 0} \alpha = 0, \ \ \lim_{k\to 0} \alpha = 1, \ \ \lim_{\alpha\to 0} \mu = k, \ \ \lim_{\alpha\to 1} \mu = c \tag{20}$$

5. Results

Numerical calculations were performed for both parameters α and μ. The results obtained for the more interesting parameter, i.e. for the order of fractional derivative α, are presented in Fig. 1 and Fig. 2. For clarity, a specific case is illustrated in Fig. 2.

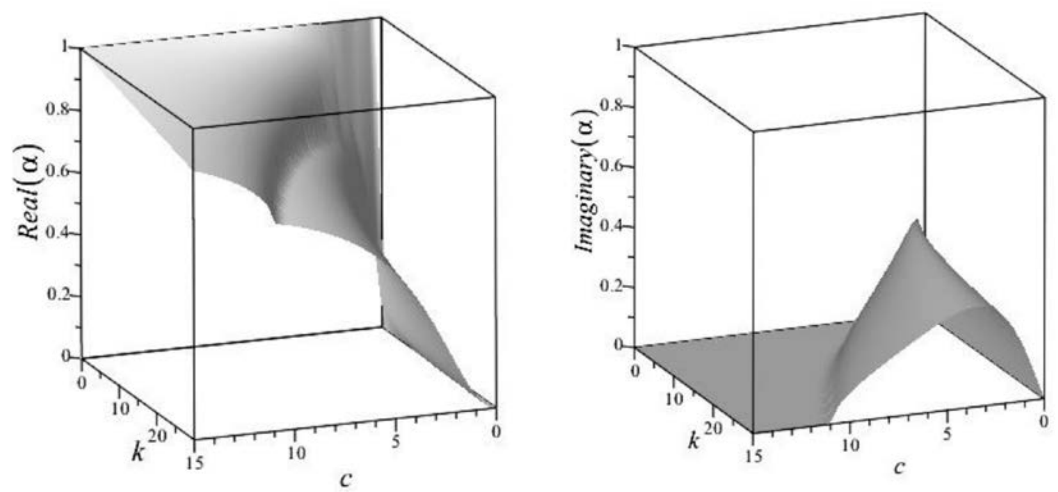

Fig. 1: The order of the fractional derivative α (calculated from formula (19), m = 1 kg).

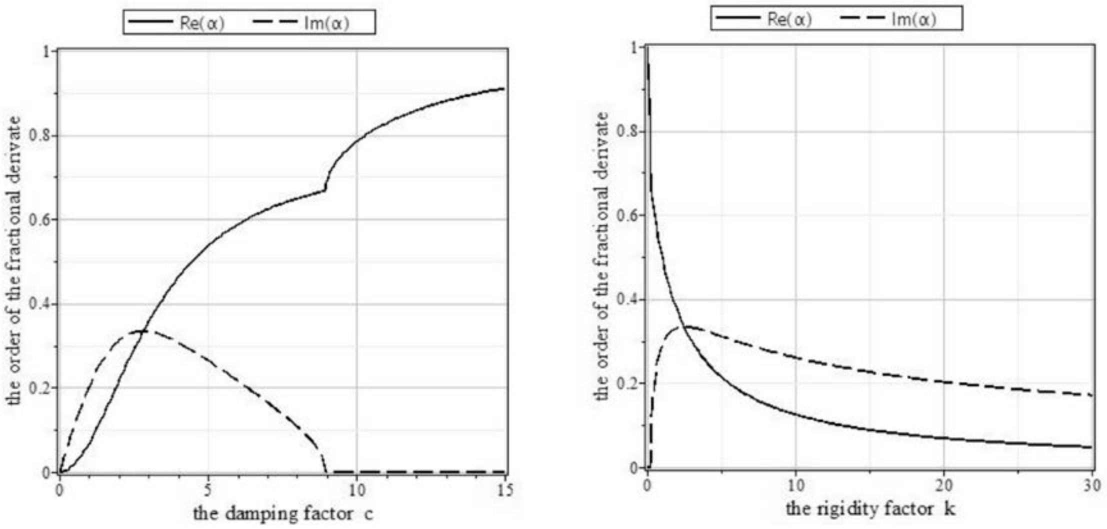

Fig. 2: The order of the fractional derivative α for selected case: k = 20 (left), c = 1 (right).

It was found that:

- the parameters of the fractional equation (4), α and μ, for which its solution is equal to the solution of the classical equation, can be calculated and the formulas are known;
- the classical equation can be replaced with the fractional one;
- excellent agreement of both equations exists only for the complex α (numerical observations).

Furthermore, the authors propose some physical interpretation of the fractional derivative in equation (4). Up to now, scientists have treated damping and rigidity in solids as two different physical phenomena. We suggest that they should be looked at as two aspects of one physical phenomenon, which can be described just by means of the fractional derivative.

6. Conclusions

This paper has considered the fractional equation for the damped harmonic oscillator in the form proposed by Herrmann (2014). The analytical solution of the fractional equation has been calculated and compared with the solution of the classical equation. The conditions of the agreement of the solutions have been calculated and the order of the fractional derivative has been derived. It can be concluded that the classical equation can be replaced with the fractional one. We propose a certain physical interpretation of the fractional derivative in the fractional differential equation for the damped harmonic oscillator.

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