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AN EXAMPLE OF OPTIMIZATION PROBLEM IN TRANSPORT

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Abstract: *This paper analyzes the problem of profit optimization known in literature for a random variable with an increasing function of increasing failure rate. The problem is generalized in case the random variable has a unimodal failure rate function. This result is illustrated by a numerical example. In the example, the unimodal failure rate function is formed from a mixture of exponential distribution and a distribution with a quadratic and increasing failure rate intensity.*

Keywords: optimization problem, ageing class, failure rate function, reliability function, unimodal function

1. Introduction

Due to the complexity of the modeled transport processes and systems, there is a need for use of appropriate methods and tools ensuring effective realization of research and analysis of results obtained. Depending on the kind of analyzed research problems, appropriate methods of delineating optimal and quasi-optimal solutions were implemented and criterial functions, e.g.: the reliability and the average profit from the work of a technical object (Knopik et al., 2016), the unit cost of preventive replacement (Knopik et al., 2017), the availability and the profit per unit time (Knopik et al., 2018 and Migawa et al., 2016). This paper examines a problem of profit optimization, which is associated with the contracting of transport services. In particular, a profit optimization model resulting from the relationship between the forwarder and the carrier is analyzed. The problem of optimization presented in the work is a generalization of the problem known from literature (Brusset, 2010 and Lariviere et al., 2001). In the paper (Brusset, 2010) it is assumed that the time of carrying out the transport task is a random variable with a increasing failure rate function (*IFR*). This paper shows that the theses of the paper (Brusset, 2010) are true for the wider class of probability distributions than the *IFR*. The review of classes and their properties is included in the paper (Lariviere, 2006). In the papers (Knopik, 2005 and Knopik, 2006 and Knopik, 2010), properties of a class of distributions useful for technical objects operation are examined. T stands for a non-negative random variable with $f(x)$ density, $F(x)$ distribution function, $R(x)$ reliability function and $\lambda(x)$ failure rate function. *IFR* is used to denote a class of non-negative random variables with a non-decreasing $\lambda(x)$ failure rate function. The paper (Brusset, 2010) analyzes the problem of criteria function optimization in the following form:

$$g(x) = \alpha F(x) + xR(x) \tag{1}$$

where $\alpha \geq 0$ is a real number.

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The first order of derivative of the function $g(x)$ has the following form:

$$g'(x) = f(x)(\alpha - x) + R(x) \quad (2)$$

The second order of derivative may be written as:

$$g''(x) = f'(x)(\alpha - x) - 2f(x) \quad (3)$$

For the derivative $g''(x)$ to exist, it is sufficient that the derivative $f'(x)$ exists. The following conclusions include the basic properties of $g(x)$ function and its derivatives.

Conclusion 1. $g(0) = 0$, $g(\infty) = \alpha$.

Conclusion 2. $g'(0) = \alpha f(x) + R(x)$, $g'(\alpha) = R(\alpha) \geq 0$, $g'(\infty) = 0$.

Conclusion 3. $g''(\alpha) = -2f(x) \leq 0$.

Conclusion 4. If $0 \leq x \leq \alpha$, then the criteria function $g(x)$ increases.

From conclusion 3 it follows that if there is a point x_0 such that $f'(x_0) = 0$, then the criterion function $g(x)$ reaches the local maximum at x_0. From conclusion 4 it can be concluded that if there is x_0 such that $f'(x_0) = 0$, then $x_0 > \alpha$.

The equation $g'(x) = 0$ may be written as:

$$\lambda(x) = \frac{1}{x-\alpha} \quad (4)$$

The right side of the equation is an increasing function, the left side is a decreasing function. This fact allows you to arrive at an important conclusion:

Conclusion 5. If $T \in IFR$, then criteria function $g(x)$ has exactly one maximum.

Proof. Equation (4) has exactly one solution x_0, thus on the basis of conclusions $g'(x_0) = 0$ i $g''(x_0) < 0$. It follows that the criteria function $g(x)$ reaches exactly one maximum.

2. Criteria function for unimodal failure rate function

Examples of functions from the *IFR* class, for which the function $g(x)$ reaches the maximum are given in the paper (Brusset, 2010). Reliability theory analyzes a class of technical objects lifetime distribution much wider than the *IFR* class. Such lifetimes include lifetimes with a unimodal failure rate function. A set of lifetimes is built below such that their failure rate functions are unimodal and the criteria function $g(x)$ reaches its exactly one maximum.

To this end, the following symbols are introduced:

x_0 – point, at which failure rate function $\lambda(x)$ reaches maximum,

x_1 – lowest root of equation (4),

g – limit value for failure rate function $\lambda(x)$, $\lim_{x\to\infty} \lambda(x) = g$,

x_2 – value of variable *x*, for which function $g_1(x) = \frac{1}{x-\alpha}$ equals g, which means that $x_2 = \alpha + \frac{1}{g}$.

Since function $g_1(x) > 0$ for $x > \alpha$ it follows that $x_1 > \alpha$.

We shall prove that, taking into consideration certain assumptions for $x \in (x_1, x_2)$, the equation (4) has no roots. This is equivalent to the criterion function $g(x)$ reaching exactly one maximum.

Conclusion 6. If $\lambda(x)$ is unimodal and for $x \in (x_1, x_2)$ the following $\lambda'(x)(x-\alpha)^2 + 1 > 0$ is valid, then the criteria function $g(x)$ has exactly one maximum.

Proof. Let $v(x) = \lambda(x) - \frac{1}{x-\alpha}$. It is known that $v(x_1) = 0$. The derivative $v'(x)$ may be written as: $v'(x) = \lambda'(x) - \frac{1}{(x-\alpha)^2}$. On the basis of the assumption $\lambda'(x)(x-\alpha)^2 + 1 > 0$ a conclusion is formed that $v'(x) > 0$. Thus the function $v(x)$ increases for $x \in (x_1, x_2)$. For $x \geq x_2$ it is clear that $v(x) > 0$. Therefore, the equation (4) has no roots for $x \in (x_1, x_2)$, the function $g(x)$ reaches exactly one maximum.

3. Numerical example

Below is an example of the failure rate function for lifetime T, the distribution of which is a mixture of exponential distribution and a distribution with increasing quadratic failure rate function: $\lambda_1(x) = ax^2 + bt + c$. An extensive review of mixture results is included in paper (Lai et al., 2006). In particular, that paper presents the results concerning the possible shapes of failure rate function for pairs of different probability distributions. Variables T_1 and T_2 have probability densities $f_1(x)$ and $f_2(x)$, distribution functions $F_1(x)$ and $F_2(x)$, reliability functions $R_1(x)$ and $R_2(x)$. The reliability function $R(x)$ for the mixture of variable distributions T_1 and T_2 is expressed in the formula:

$$R(x) = pR_1(x) + (1-p)R_2(x) \tag{5}$$

where p is such mixture ratio $0 \le p \le 1$.

The failure rate function of the mixture can be written as the following mixture:

$$\lambda(t) = w(t)\lambda_1(t) + (1-w(t))\lambda_2(t) \tag{6}$$

where $w(t) = \frac{pR_1(t)}{R(t)}$.

Proposition 1: For the first derivative of $w(t)$, we have:

$$w'(t) = w(t)\big(1-w(t)\big)\big(\lambda_2(t) - \lambda_1(t)\big)$$

Proposition 2: The first derivative of $\lambda(t)$ is:

$$\lambda'(t) = \big(1-w(t)\big)\left[\left(-w(t)\big(\lambda_2(t)-\lambda_1(t)\big)^2 + \lambda'_2(t)\right) + w(t)\lambda'_1(t)\right]$$

Proposition 3: If $\lambda_1(t) = \lambda$, then:

$$\lambda'(t) = \big(1-w(t)\big)\big(-w(t)(\lambda_2(t)-\lambda)^2 + \lambda'_2(t)\big)$$

Let $\lambda_1(t) = \lambda$, $\lambda_2(t) = at^2 + bt + c$, $g_1(t) = w(t)(at^2 + bt + c - \lambda)^2$, $g_2(t) = 2at + b$, where $a > 0, b \ge 0, c \ge 0$.

The equation $\lambda'(t) = 0$ is equivalent to the equation:

$$g_1(t) = g_2(t) \tag{7}$$

For the ratio $u(t) = \frac{g_1(t)}{g_2(t)}$, we have:

$$\lim_{t\to\infty} u(t) = \infty \tag{8}$$

For the first derivative of $u_1(t) = \frac{(\lambda_2(t)-\lambda)^2}{g_1(t)}$, calculate:

$$u'_1(t) = \frac{2(at^2+bt+c-\lambda)}{(2at+b)^2}\big(3a^2t^2 + 3abt + b^2 - a(c-\lambda)\big) \tag{9}$$

If $c = \lambda$, then $u(t) = u_1(t)w(t)$ increasing from $u(0) = 0$ to $u(\infty) = \infty$ and the equation $u(t) = 1$ has exactly one solution, then $\lambda(t)$ is *UBT*.

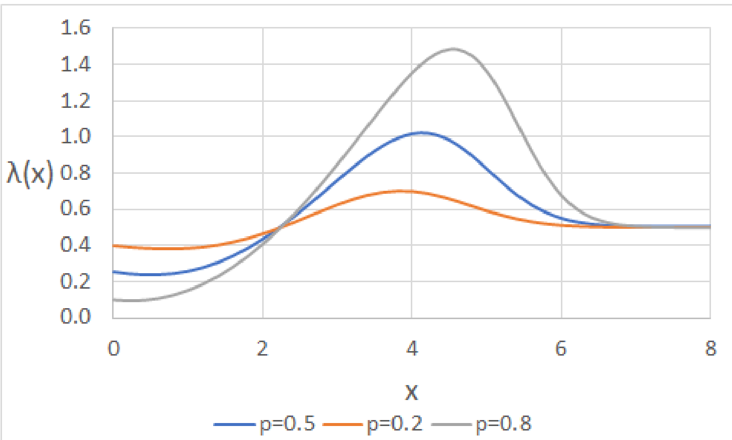

Fig. 1: Graphs of failure rate function for $p \in \{0.2, 0.5, 0.8\}$ *as well as* $a = 1, b = 0, c = 0, \lambda_2 = 0.5$.

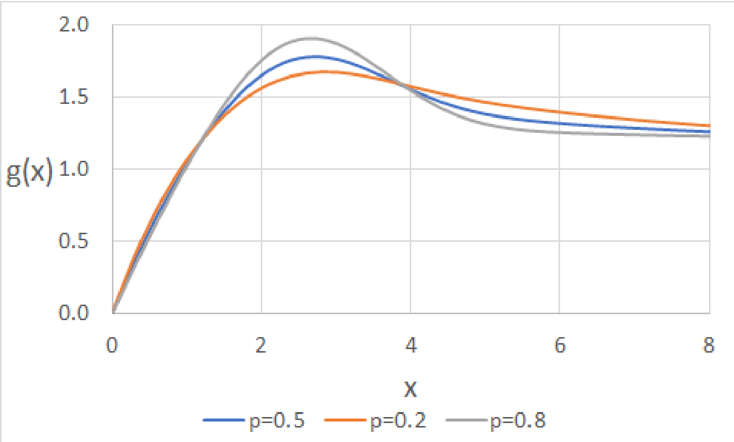

Fig. 2: Graphs of criteria functions for $p \in \{0.2, 0.5, 0.8\}$ *as well as* $a = 1, b = 0, c = 0, \lambda_2 = 0.5$.

Figure 1 shows three unimodal failure rate functions $\lambda(t)$ and figure 2 shows three related realizations of criteria functions $g(x)$, for $p \in \{0.2, 0.5, 0.8\}$ as well as $a = 1, b = 0, c = 0$ and $\lambda_2 = 0.5$. For analyzed case, the criteria function $g(x)$ reaches exactly one maximum. The values of maximum criteria function depend on mixture parameter p.

4. Conclusions

This paper analyzes an economic optimization problem previously formulated in literature. The properties of the criterion function were examined, showing that the criteria function reaches an exactly one maximum for a much widely class of probability distributions than in the quoted literature. The paper concludes with an analysis of the generalizations of the optimization problem when the failure rate function is unimodal. In this paper, the probability distribution with the unimodal rate function was obtained from a mixture of an exponential distribution and a distribution with a quadratic and increasing failure rate function.

References

Brusset, X. (2010) Modeling contractual relationships in transport. Ph.D.thesis. Louvain School of Management. Place des doyens, 1, B-1348, Louvain la Neuve, Belgium.

Knopik, L. (2005) Some results of a class of lifetime distributions. Control and Cybernetics, 35(2), pp. 1-8.

Knopik, L. (2006) Characterization of a class of lifetime distribution. Some results of a class of lifetime distributions. Control and Cybernetics, 34(1), pp. 407-414.

Knopik, L. (2010) Method of choice efficiency strategy of the maintenance of technical object. University of Science and Technology in Bydgoszcz, Publishing Department, Dissertations no. 145.

Knopik, L., Migawa, K. and Wdzięczny, A. (2016) Profit optimalization in operation systems. Polish Maritime Research, vol. 23, no. 1(89), pp. 93-98.

Knopik, L. and Migawa, K. (2017) Optimal age-replacement policy for non-repairable technical objects with warranty. Eksploatacja i Niezawodnosc – Maintenance and Reliability, vol. 19, no. 2, pp. 172-178.

Knopik, L. and Migawa, K. (2018) Multi-state model of maintenance policy. Eksploatacja i Niezawodnosc – Maintenance and Reliability, vol. 20, no. 1, pp. 125-130.

Lai, Ch. D. and Xie, M. (2006) Stochastic ageing and dependence for reliability. Springer, New York.

Lariviere, M. A. (2006) A note probability distributions with increasing generalized failures rates. Operations Research 54(3), pp. 602-604.

Lariviere, M. A. and Proteus, E. L. (2001) Selling to the newsvendor: An analysis of price-only contract. Manufacturing & Service Operations Management, 3(4), pp. 293-305.

Migawa, K., Knopik, L. and Wawrzyniak, S. (2016) Application of genetic algorithm to control the availability of technical systems. Engineering Mechanics 2016, IT AS CR, Prague, pp. 386-389.