doi: 10.21495/71-0-105

25th International Conference
ENGINEERING MECHANICS 2019
Svratka, Czech Republic, 13 – 16 May 2019

# LOCAL STABILIZATION OF THE QUASIPERIODIC RESPONSE OF THE GENERALIZED VAN DER POL OSCILLATOR

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**Abstract:** *The generalized van der Pol equation is often used for description of various effects originating in the aero-elasticity of large slender engineering structures. This applies mainly to the quasiperiodic beatings that can be encountered especially in lock-in regimes when the vortex frequency becomes close to the structure eigenfrequency with a small detuning. The current paper presents numerical analysis of influence of the sub- or superharmonic excitation on the character of the response of a generalized van der Pol oscillator. This way it complements two previous papers of the authors dealing with stability analysis of certain types of the stationary periodic or quasiperiodic response of the system under study.*

**Keywords: Generalized van der Pol equation, Quasiperiodic response, Sub- and Superharmonic synchronization, Beating effect, Numerical simulation.**

## 1. Introduction

Interaction of the air stream and large slender engineering structures gives rise to a wide spectrum of the non-linear aero-elastic processes. Namely the beating effects emerging in relation with vortex shedding represent a category which is dangerous of the functionality and safety of the structure. This applies mainly to the quasiperiodic beatings that can be encountered especially in lock-in regimes when the vortex frequency becomes close to the structure eigenfrequency $\omega_0$ with a small detuning. However, due to non-linear character of the underlying physical system, the effect of sub- or superharmonic synchronization can be encountered. In this mode also the close proximity of integer multiples or fractions of the driving frequency and the eigenfrequency of the structure can cause undesired or even dangerous effects. The quasiperiodic phenomena of the basic aero-elastic model resonance and their stability properties were theoretically investigated by the authors in the past (Náprstek and Fischer, 2018a). The recent study of the authors (Náprstek and Fischer, 2018b) concentrates on the stability assessments of the sub- or superharmonic synchronization and its effect on the free component of the system response. The both works use the harmonic balance method for analytical investigation and their results depend on the fulfilment of the relevant assumptions. The current paper present selected and illustrative results of a thorough numerical study of the same model subjected to a wide range of excitation modes covering the subharmonic, resonant and superharmonic excitation.

The commonly used Single-Degree-of-Freedom (SDOF) or the more complicated Two-Degree-of-Freedom (TDOF) section models of a structure in the air stream represent a reasonable compromise between complexity and ability to characterise the dynamic processes. Such type of models is used often in the aerodynamic wind tunnel experiments and well serve their purpose. However, it appears that in many cases when the TDOF model is used, one of the components is dominant and, thus, the second one can be neglected. It reveals that majority of the resulting SDOF systems can be modelled by the van der Pol-Duffing or generalized van der Pol type equations or their combination adjusting degree of individual non-linear terms or their coefficients. This hypothesis was confirmed many times and is generally accepted, see, e.g., (Koloušek et al., 1984).

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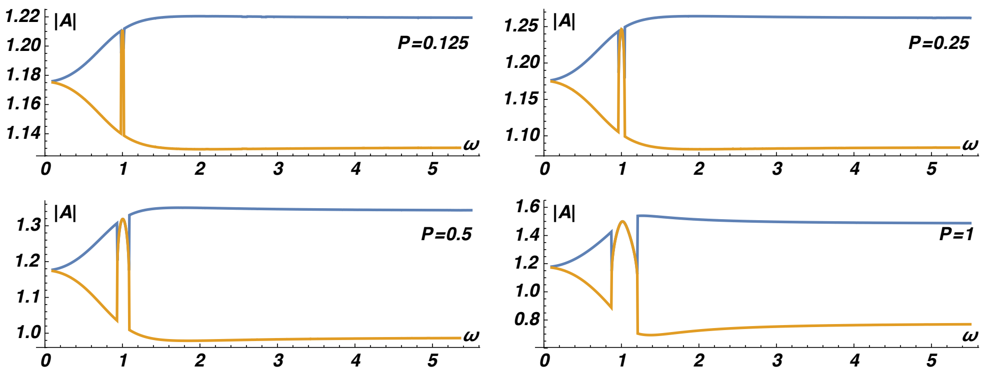


*Fig. 1: Courses of maximal (upper curve, blue) and minimal (lower curve, yellow) amplitudes $A$ given by the harmonic balance method in Eq. (3) for four values of the excitation amplitude $P = 0.125, 0.25, 0.5, 1$.*

## 2. Mathematical model

The behaviour of the structure in the stream can be characterized by means of the generalized van der Pol equation with a harmonic right hand side. The generalization consists in an inclusion of the fourth order term in the damping. This way it is possible to describe both of the most important limit cycles, where the first is stable (attractive) and the second unstable (repulsive). Consequently, the governing equation reads:

$$\ddot{u} - (\eta - \nu u^2 + \vartheta u^4)\dot{u} + \omega_0^2 u = P\omega^2 \cos\omega t \tag{1}$$

where $u$ is the response of the system; $\omega_0^2 = K/m$ is the eigenfrequency of the associated linear system with stiffness $K$ and concentrated mass $m$; $\eta, \nu, \vartheta$ are positive coefficients of linear viscous and non-linear damping; $\omega^2 P$ is the amplitude of the harmonic excitation (excitation force per unit mass, frequency $\omega$). $P$ can be interpreted as an amplitude of pressure variation during vortex shedding.

Supposing the stationary response, i.e., the state when the transient time elapses and the influence of initial conditions vanishes, the response can be expected to have the harmonic form with slowly varying amplitude $U$ and phase shift $\varphi$:

$$u = U\cos(\omega t + \varphi)\,. \tag{2}$$

Employing the harmonic balance procedure, the differential system for $U$ and $\varphi$ as functions of the "slow time" can be deduced:

$$\dot{A} = \frac{\eta}{2}A(1 - A^2 + 2\gamma A^4) - \frac{\eta}{2}Q\omega\sin\varphi, \tag{3a}$$

$$\dot{\varphi} = \Delta - \frac{\eta}{2A}Q\omega\cos\varphi, \tag{3b}$$

where

$$\Delta = \frac{\omega_0^2 - \omega^2}{2\omega} \approx \omega_0 - \omega, \quad A^2 = \frac{1}{4}\frac{\nu}{\eta}U^2, \quad \gamma = \frac{\vartheta\eta}{\nu^2}, \quad \delta^2 = \frac{\Delta^2}{\eta^2}, \quad Q^2 = \frac{\nu P^2}{4\eta^3}.$$

For the detailed derivation of Eq. (3), the reader is kindly referred to the earlier publication (Náprstek and Fischer, 2018a). It is clear from the structure of the harmonic balance procedure that the average amplitudes given by Eq. (3) cannot reflect higher harmonic modes than that given by the ansatz Eq. (2), cf. Fig. 1.

It suggests itself to use the same technique also for identification the sub- and superharmonic properties of the response. However, it appears that the straightforward application of the harmonic balance procedure does not lead to relevant results in this case. Indeed, assuming generalization of the Eq. (2):

$$u = U\cos(n\omega t + \varphi)\,. \tag{4}$$

for $n = 2, 3\ldots$ and $n = 1/2, 1/3, \ldots$, the harmonic balance leads to a set of identical expressions for the amplitude and slightly different relations for the phase shift. Regarding $A$, the expression common for all cases represents the asymptotically constant amplitude $A$ for $\gamma <= 1/8$ and diverging otherwise and reads

$$\dot{A} = \frac{\eta}{2}A(1 - A^2 + 2\gamma A^4)\,. \tag{5}$$

The resulting formulas for the phase shift $\varphi$ for all cases indicate that $\varphi$ are evenly increasing in time with velocities depending on $n$. The individual expressions are given in Table 1.

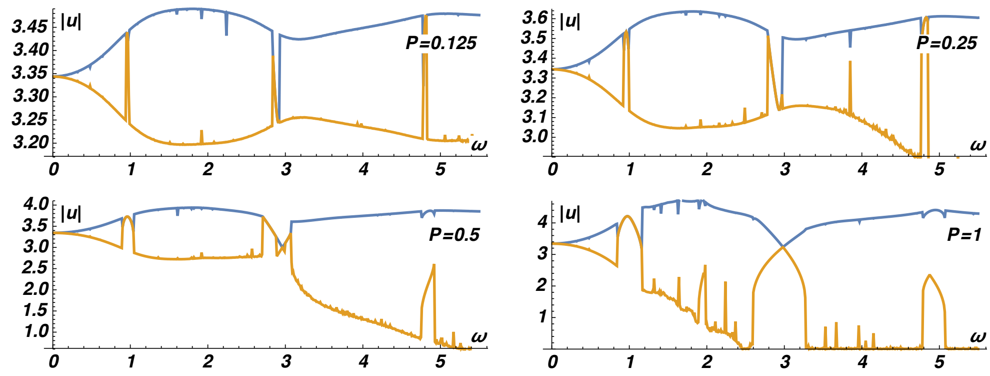


*Fig. 2: Numerically obtained resonance curves of the generalized van der Pol equation Eq. (1) for four values of the excitation amplitude* $P = 0.125, 0.25, 0.5, 1$. *Blue and yellow curves show the maximal and minimal amplitudes, respectively.*

It is clear that the apparently existing sub- or superharmonic effects (cf. Fig. 2) have to be modelled using other approaches. The theoretical aspects of this topic are in greater detail discussed in the recently submitted paper (Náprstek and Fischer, 2018b). The present work will continue by showing the problem from the numerical perspective.

## 3. Numerical evaluation

The numerically obtained (non-linear) resonance curves of the Eq. (1) for different settings of the excitation amplitude $P$ are shown in Fig. 2. Values of the other parameters used in simulations are $\eta = 1, \nu = 0.5, \vartheta = 0.025 \Rightarrow \gamma = 0.1$; $\omega_0 = 1$. The integration was performed using the $M = 2$ variant of the implicit Gear method implemented as the routine `gear2` of the GNU Scientific Software Library (Galassi et al., 2009) and as the default method of the `NDSolve` command in Wolfram Mathematica v. 11. The non-stationarity of the response is shown in each plot using two curves; the upper and the lower curve represent maximal and minimal values of the envelope of the response, respectively. The system is stationary if both curves coincide. This way it is visible in Fig. 2 that the response is stationary close to the resonance (for $\omega \approx \omega_0$) and for lower values of $P$ also for odd superharmonic frequencies, i.e., for $\omega$ below $3\omega_0$ and $5\omega_0$. The effect is better visible in the detailed plots in Fig. 3 where three stationary intervals are shown in the zoomed graphs for all driving amplitudes.

Presence of the stationary response in the vicinity of the resonance $\omega_0 = 1$ is natural and is theoretically justified in the earlier publication (Náprstek and Fischer, 2018a). Its width and location depends on parameters of the structure. Fig. 2 and the upper left plot of Fig. 3 show dependence of the excitation amplitude $P$ and width of the stationary area. Namely, the width of the stationary area increases with increasing excitation amplitude.

The assumed simple harmonic form of the solution Eq. (2) which is used in the paper (Náprstek and Fischer, 2018a) cannot comprise higher or lower harmonics of the response. However, the numerical resonance curve in Fig. 2 and the zoomed plots in Fig. 3 show changes of the response character in the neighbourhood of the superharmonic frequencies. Small irregularities are visible for $\omega = 1/2, 2$ or $4$, but the apparently stationary character of the response is present for $\omega = 3$ and $5$. With increasing excitation amplitude, the width of the stationary areas increases but at the same time their number lowers. The non-stationary response exhibits the beating effects, where the beating frequency decreases to zero in the stationary areas and increases with increasing distance of the driving frequency from the (super) resonance frequency. However, starting from a certain detuning (distance from the resonance), the numerically identified period of the response starts to behave irregularly and is hard to correctly interpret.

*Tab. 1: Values of the phase shift* $\varphi$ *for selected sub-/superharmonic components of the response*

| | $n=2$ | $n=3$ | $n=4$ | $n=1/2$ | $n=1/3$ | $n=1/4$ |
|---|---|---|---|---|---|---|
| $\dot{\varphi} =$ | $\frac{1}{2}\Delta - \frac{3}{4}\omega$ | $\frac{1}{3}\Delta - \frac{4}{3}\omega$ | $\frac{1}{4}\Delta - \frac{15}{8}\omega$ | $2\Delta + \frac{3}{4}\omega$ | $3\Delta + \frac{4}{3}\omega$ | $4\Delta + \frac{15}{8}\omega$ |

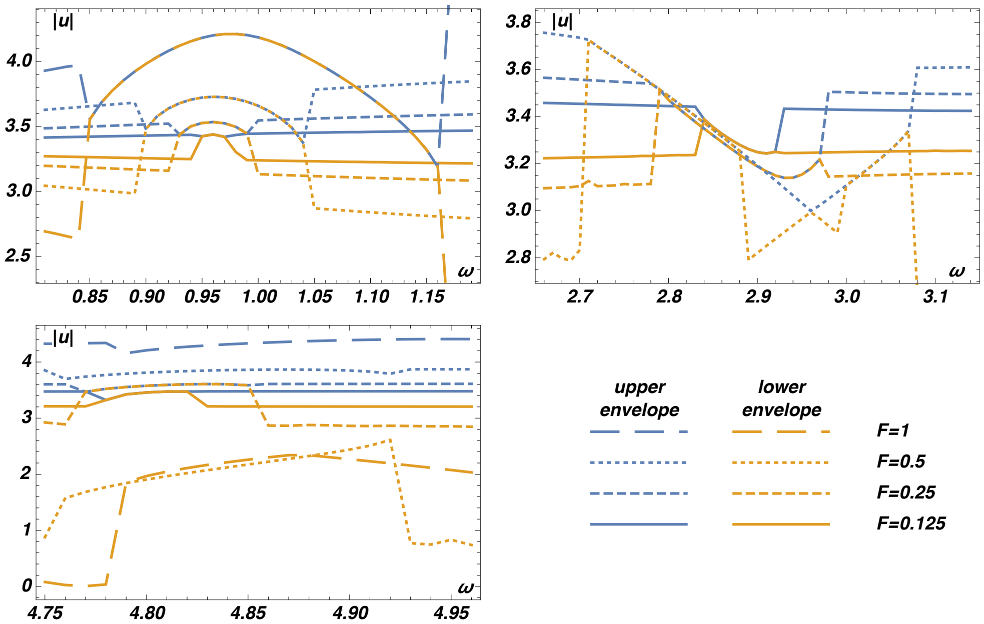


*Fig. 3: Detailed plots of the stationary zones for the resonance frequency* $\omega_0 = 1$ *(*$\omega \in (0.8, 1.2)$*, top left) and two superharmonic stationary zones:* $\omega \in (2.7, 3.1)$*, top left and* $\omega \in (4.75, 5)$*, bottom left*

## 4. Conclusions

The SDOF non-linear system described by the generalized van der Pol equation has a particular importance in the field of engineering mechanics. This equation describes the state when the total linear damping component drops below zero and only non-linear effects are able to stabilize the system. It is able to characterize the reduced flutter as one of the post-critical response types of an aero-elastic system. An attempt to describe the higher and lower harmonics of the generalized van der Pol equation using the harmonic balance procedure proved inapplicable. The numerical solution was performed and its selected resonance properties were identified as a supplement to the approximate analytical treatment which was published by the authors in the past. The response is generally characterised by the beating character, however, for certain excitation frequencies is the response stationary. The analysis in this communication is presented for a single value of the damping parameter and thus it serves for illustrative purposes only. However, it indicates open questions and necessity to use a more sophisticated approach.

## Acknowledgement

The kind support of the Czech Science Foundation project No. 19-21817S and of the RVO 68378297 institutional support are gratefully acknowledged. Access to computing and storage facilities owned by parties and projects contributing to the National Grid Infrastructure MetaCentrum provided under the programme ”Projects of Large Research, Development, and Innovations Infrastructures” (CESNET LM2015042), is greatly appreciated.

## References

Galassi, M., Davies, J., Theiler, J., Gough, B., Jungman, G., Booth, M., and Rossi, F. (2009) *GNU Scientific Library: Reference Manual*. Network Theory Ltd., 3rd edition.

Koloušek, V., Pirner, M., Fischer, O., and Náprstek, J. (1984) *Wind Effects on Civil Engineering Structures*. Elsevier, Amsterdam.

Náprstek, J. and Fischer, C. (2018a) Analysis of the quasiperiodic response of a generalized van der Pol nonlinear system in the resonance zone. *Computers and Structures*, 207, pp. 59–74. CIVIL-COMP 2017.

Náprstek, J. and Fischer, C. (2018b) Super and sub-harmonic synchronization in generalized van der Pol oscillator. *Computers and Structures*. submitted.