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TEMPORAL-SPATIAL DISPERSION ANALYSIS OF FINITE ELEMENT METHOD IN IMPLICIT TIME INTEGRATION

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Abstract: *The temporal-spatial dispersion analysis for the linear finite element method with implicit time integration is presented. The Newmark method with* $\beta = 1/2$ *and* $\gamma = 1/4$ *is used as well as the consistent mass matrix. The temporal-spatial dispersion relationships are derived in the closed form and analyzed due to errors in numerical wave speed of propagation of harmonic wave. Based on this temporal-spatial dispersion analysis, a suitable mesh size and time step size for allowed errors in phase speed are mentioned as well as we present the polar dispersion graphs.*

Keywords: Finite element method, time integration, Newmark method, dispersion errors, wave propagation.

1. Introduction

In numerical modelling of wave propagation by the finite element method ((Hughes, 2000)), both the spatial and temporal discretization introduce dispersion errors. In general, the finite element solution is polluted by dispersion errors as an effect of spatial finite element discretization and by the period elongation errors and numerical damping of the direct time integration, see (Schreyer, 1983; Mullen and Belytschko, 1982). The dispersion errors in finite element modelling are caused by the difference of numerical wave speeds from the wave speeds in continuum and are dependent on the frequency of the propagation wave, its orientation in the finite element grid, mesh size, time step size, element type, a choice of mass matrix and other numerical parameters of the method.

In this paper, these dispersion errors are studied on the example of the plane strain linear (4-noded) finite elements, see (Kolman et al, 2013, 2016). We extend the dispersion study into analysis of the finite element method in implicit time integration based on the Newmark method, see (Newmark, 1959). Moreover, we recommend a choice of numerical parameters as a mesh size and a time step size for finite element modelling with implicit time integration of elastic wave propagation in solids.

The most widely used group of one-time step methods for $implicit$ direct time integration is the Newmark family, where the approximation of the displacement and velocity vector are controlled by two parameters β and γ, for detail see (Newmark, 1959). If $\gamma < 1/2$ and $\gamma > 1/2$ a negative and positive damping is introduced by the algorithm, respectively. For this reason, the further analysis is restricted only to $\gamma = 1/2$. Then this general group includes several well-known cases, such as the average acceleration method with $\beta = 1/4$, the linear acceleration method with $\beta = 1/6$ and the Fox-Goodwin method with $\beta = 1/12$. The explicit central difference method can be incorporated with $\beta = 0$. For more details, see (Hughes, 2000).

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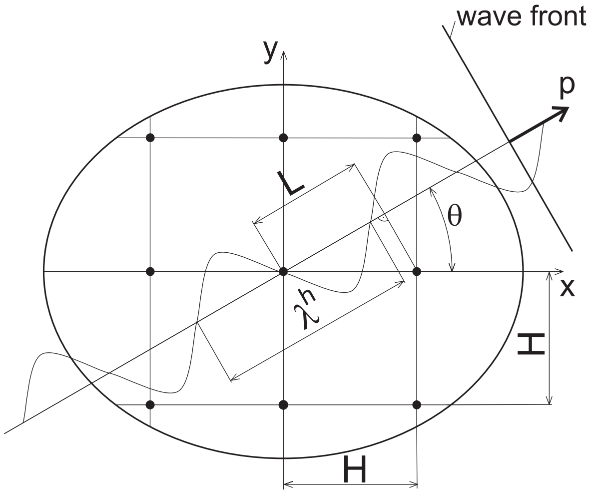

Fig. 1: Two-dimensional infinite bilinear regular finite element mesh and plane wave inclined by angle θ.

2. Temporal-spatial dispersion analysis

There are several methods available to compute the dispersive behavior and all of them are based on the analysis of the harmonic wave propagated in periodic space and discrete time and all of the use the Fourier method. The solution is then in form of Bloch-plane wave.

The nodal displacements at the time $t = s\Delta t$ of a plane wave problem, see Fig. 1(a), corresponding to the wave solution in discrete form, are prescribed in the form

$$\begin{aligned} u^h_{m,n} &= U_{0\,m,n} e^{i(k^h x_m p_x + k^h y_m p_y - \omega s \Delta t)} \\ v^h_{m,n} &= V_{0\,m,n} e^{i(k^h x_m p_x + k^h y_m p_y - \omega s \Delta t)} \end{aligned} \tag{1}$$

where $u^h_{m,n}$ and $v^h_{m,n}$ are the displacements in x and y direction in nodes (m, n) with the coordinates x_m and y_m, see Fig. 1(b), $U_{0\,m,n}$ and $V_{0\,m,n}$ are unknown amplitudes, ω marks the angular velocity, Δt is the time step size, k^h is the numerical wave number and p_x and p_y are components of the unit normal vector expressed as $p_x = \cos\theta$ and $p_y = \sin\theta$, where angle θ defines the direction of wave propagation through the mesh. After these wave motion is applied on the equation of discretized system and grid dispersion behavior can be obtained, see (Kolman et al, 2013).

Using the relations for the displacement, velocity and the acceleration approximations of the Newmark method in the matrix form of equation of motion, we obtain the final system of equation for the periodic part of the problem. Then the dispersion relation can be obtained from the eigenfrequency analysis of that system. The angular velocities ω_i of the propagated waves expressed in terms of β for $\gamma = 1/2$ are

$$\omega_i = \frac{2}{\Delta t} \arcsin \sqrt{\frac{\Lambda_i E \Delta t^2}{4\beta\Lambda_i E \Delta t^2 + 4H^2\rho}}, \qquad i = 1, 2, ..., N_c \tag{2}$$

where E is the Young modulus, ρ is the density of the material and H is the size of the element. The eigenvalues Λ_i are obtained from the generalized eigenvalue problem of one periodic cell containing one corner node for bilinear element, see (Kolman et al, 2013).

This dispersion relation can be transformed in dimensionless form by setting $\bar{\omega}_i = \omega_i H/c_1$ for dimensionless angular velocity (c_1 is the velocity of the longitudinal wave propagating in isotropic elastic domain, $c_1 = \sqrt{(\Lambda + 2G)/\rho}$), Λ and G are Lamé's coefficients, by denoting the Courant number $C = \Delta t c_1/H$ and denoting $c_0 = \sqrt{E/\rho}$

$$\bar{\omega}_i = \frac{2}{C} \arcsin \sqrt{\frac{\Lambda_i C^2 c_0^2}{4\beta\Lambda_i C^2 c_0^2 + 4c_1^2}}, \qquad i = 1, 2, ..., N_c \tag{3}$$

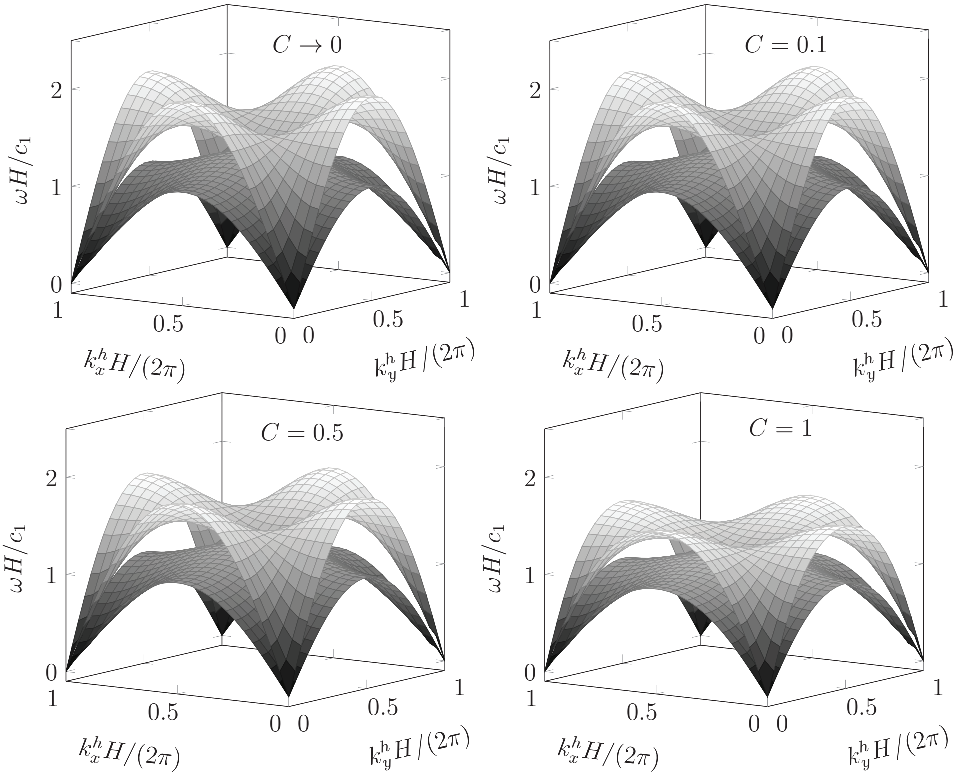

Fig. 2: Dispersion curves plotted over the first Brillouin zone of a linear 4-noded plain strain element for several Courant numbers (representing a non-dimensional time step).

Such relation serves for setting the time step size and mesh size for required accuracy in finite element modelling of elastic wave propagation due to dispersion. N_c is the number of dispersion branches.

Results for the linear elements, the average acceleration methods with $\beta = 1/2$ and $\gamma = 1/4$ and the consistent mass matrix are shown in Fig. 2 and 3 for different values of the Courant number C. The dispersion curves in Fig. 2 show that the angular velocities of the propagating wave through the periodic finite element structure are limited and sensitive on the value of the time step size.

The suitable mesh size, measured by finite element edge H, of the linear plain strain finite element was established in (Kolman et al, 2013). Expecting the spatial dispersion errors approximately 2%, the size of a linear elements with respect to the minimum wave length of the propagated wave λ is approximately $H/\lambda = 1/10$. Based on the dispersion graphs presenting in this paper, setting of the time step size Δt for linear elements measured by the Courant number is recommended as $C = 0.5$.

3. Conclusions

In this paper, we have presented the temporal-spatial dispersion analysis of the finite element method with bilinear shape functions. The analysis has been performed for the plane problem, the consistent mass matrix and the implicit time integration based on the Newmark method with $\beta = 1/2$ and $\gamma = 1/4$. Based on this study, we are able to recommend the setting of the mesh size defined as the element edge $H \leq 10\lambda$, where λ is the wavelength of propagating wave through the finite element mesh. The time step is recommended to choice as $\Delta t = 0.5H/c_1$, where c_1 is the speed of longitudinal wave. We tested this output of the dispersion analysis by several numerical tests. In the future, we plan to extent the analysis for other modern implicit time integrators as the generalized-α method or the Bathe method.

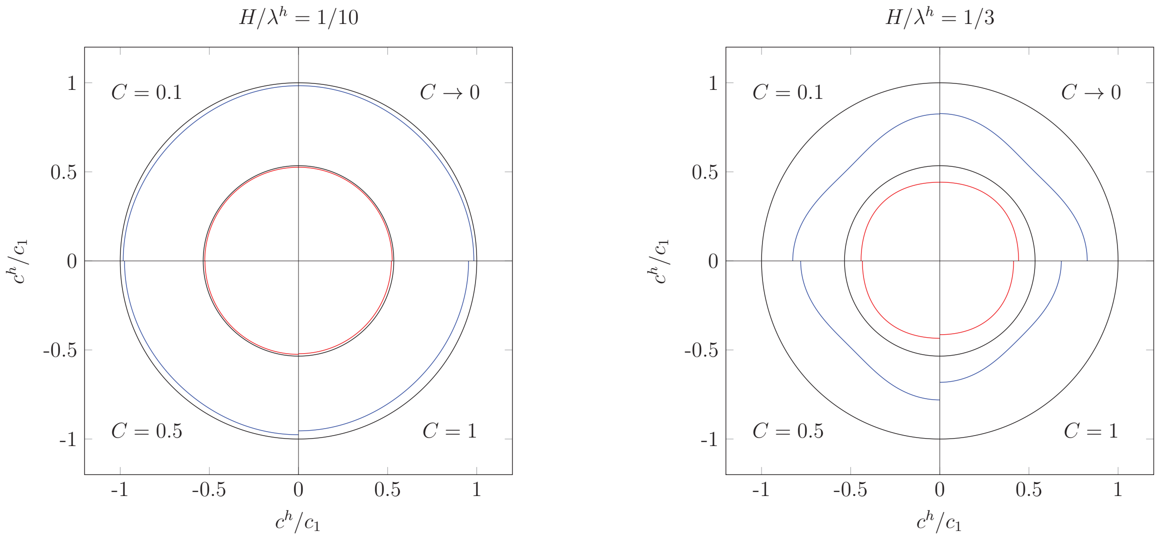

Fig. 3: Polar diagrams for two-dimensional infinite bilinear 4-noded regular finite element mesh for $H/\lambda = 1/10$ *(on the left) and* $H/\lambda = 1/3$ *(on the right).*

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