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APPLICATION OF INTERVAL ARITHMETIC IN HEAT TRANSFER MODELLING IN BIOLOGICAL TISSUE DOMAIN

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Abstract: *In the paper, the description of a heat transfer process proceeding in a one-dimensional, non-homogeneous biological tissue domain is presented. The paper concerns imprecisely defined transient bio-heat transfer problems, when in the mathematical model the uncertain parameters are defined and treated as interval numbers. The base of mathematical model is given by the Pennes interval set of equations supplemented by the adequate boundary-initial conditions. The problem discussed is solved with the use of the interval version of the finite difference method applying classical and directed interval arithmetic rules. In the final part of the paper the examples of numerical simulations are shown, in particular the comparison of both types of interval arithmetic.*

Keywords: Bio-heat transfer, Interval finite difference method, Interval numbers, Directed interval arithmetic, Classical interval arithmetic.

1. Introduction

Thermophysical parameters of biological tissue such as thermal conductivity, volumetric specific heat and perfusion coefficient are individual personal traits and this fact suggests the application of interval arithmetic methods at the stage of numerical modelling of the skin tissue heating process. In the paper the bio-heat transfer proceeding in a one-dimensional skin tissue domain with the interval thermophysical parameters occurring in the mathematical model is considered.

The solution of the presented problem has been obtained with the use of interval finite difference method and the rules of classical and directed arithmetic. The main advantage of the directed interval arithmetic upon the usual interval arithmetic is that the obtained temperature intervals are much narrower and their width does not increase over time (Markov, 1995). In the final part of the paper the examples of numerical computations are presented.

2. Interval governing equations

The heat transfer proceeding in the heterogeneous skin tissue domain can be described by the system of interval equations (Mochnacki, 2013 and Mochnacki, 2016)

$$L_{e-1} < x < L_e : \ \bar{c}_e \frac{\partial T_e(x,\ t)}{\partial t} = \bar{\lambda}_e \frac{\partial^2 T_e(x,\ t)}{\partial x^2} + \bar{Q}_e(x,\ t) \tag{1}$$

where e = 1, 2, 3 corresponds to the successive layers of skin such as epidermis, dermis, hypodermis, $\bar{c}_e$ is the interval volumetric specific heat, $\bar{\lambda}_e$ is the interval thermal conductivity, $\bar{Q}_e$ is the capacity of fuzzy internal heat sources, T_e is the temperature, x and t denote spatial co-ordinate and time.

The capacity of interval internal heat sources is a sum of two components

$$\bar{Q}_e(x,\ t) = G_{Be}\, c_B [T_B - T_e\,(x, t)] + \bar{Q}_{me} \tag{2}$$

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where G_{Be} is the perfusion coefficient, c_B is the volumetric specific heat of blood, T_B is the arterial blood temperature and $\bar{Q}_{me}$ is the interval metabolic heat source.

The interval equations (1) must be supplemented by the boundary-initial conditions

$$\begin{cases} x=0: \quad \bar{q}(x,t) = -\bar{\lambda}_1 \dfrac{\partial T_1}{\partial n} = \bar{q}_b \\ x=L_3: \quad \bar{q}(x,t) = -\bar{\lambda}_3 \dfrac{\partial T_3}{\partial n} = 0 \\ t=0: \quad T_e(x,0) = T_0 \end{cases} \tag{3}$$

where $\bar{q}_b$ is the given interval external heat source, T_0 is the initial temperature. Between the successive sub-domains the continuity condition is taken into account (Mochnacki, 2013).

2.1. The interval finite difference method

The interval finite difference method with the rules of directed and classical interval arithmetic is applied (Mochnacki, 2013 and Mochnacki, 2016). The time grid with a constant step $\Delta t = t^{f} - t^{f-1}$ and the geometrical mesh have been introduced.

The left-hand side of the energy equations (1) for the time t^f can be substituted by a differential quotient

$$\left(\bar{c}_e \frac{\partial \bar{T}_e(x,\ t)}{\partial t} \right)_i^{f-1} = (\bar{c}_e)_i^{f-1} \frac{(\bar{T}_e)_i^f - (\bar{T}_e)_i^{f-1}}{\Delta t} \tag{4}$$

while the right-hand side of the energy equations (1) can be transformed using the following formula

$$\left(\bar{\lambda}_e \frac{\partial^2 \bar{T}_e(x,\ t)}{\partial x^2} \right)_i^{f-1} = (\bar{\lambda}_e)_i^{f-1} \frac{(\bar{T}_e)_{i+1}^{f-1} - 2(\bar{T}_e)_i^{f-1} + (\bar{T}_e)_{i-1}^{f-1}}{(\Delta x_e)^2} \tag{5}$$

where Δx_e is the mesh step, $\bar{T}_e$ is the temperature interval and i is the index of the central point of star (Mochnacki, 1995).

The following interval differential equations are obtained

$$\begin{gathered} (\bar{T}_e)_i^f = \left(1 - 2\frac{(\bar{\lambda}_e)_i^{f-1}\Delta t}{(\bar{c}_e)_i^{f-1}(\Delta x_e)^2} \right)(\bar{T}_e)_i^{f-1} + \frac{(\bar{\lambda}_e)_i^{f-1}\Delta t}{(\bar{c}_e)_i^{f-1}(\Delta x_e)^2}\left[(\bar{T}_e)_{i+1}^{f-1} - (\bar{T}_e)_{i-1}^{f-1} \right] + \\ \frac{\Delta t}{(\bar{c}_e)_i^{f-1}} \left\{ (G_{Be})_i^{f-1} c_B \left[T_B - (\bar{T}_e)_i^{f-1} \right] + (\bar{Q}_{me})_i^{f-1} \right\} \end{gathered} \tag{6}$$

The system of equations (6) has been solved using the rules of interval arithmetic (classical and directed) and the assumption of the stability condition for explicit differential scheme.

2.2. The interval arithmetic

As first the definition of the classical interval number is introduced. Let us consider an interval $\bar{a}$ which can be defined as $\bar{a} = [a^-,\ a^+] := \{a \in \mathbf{R} | a^- \le a \le a^+\}$, where a^-, a^+ denote the beginning and the end of the interval, respectively (Neumaier, 1990 and Piasecka-Belkhayat, 2008). In the set of classical interval numbers the basic mathematical operations for $\bar{a}, \bar{b} \in \mathbf{R}$ can be defined as follows

$$\bar{a} + \bar{b} = \left[a^- + b^-,\ a^+ + b^+ \right] \tag{7}$$

$$\bar{a} - \bar{b} = \left[a^- - b^+,\ a^+ - b^- \right] \tag{8}$$

$$\bar{a} \cdot \bar{b} = [\min(a^- \cdot b^-, a^- \cdot b^+, a^+ \cdot b^-,\ a^+ \cdot b^+), \max(a^- \cdot b^-, a^- \cdot b^+, a^+ \cdot b^-,\ a^+ \cdot b^+)] \tag{9}$$

$$\bar{a}/\bar{b} = \bar{a} \cdot 1/\bar{b}, \quad 0 \notin \bar{b} \tag{10}$$

Let us consider a directed interval number $\vec{d}$ which can be defined as a set **D** of all directed pairs of real numbers $\vec{d} = [d^-, d^+] := \{d \in \mathbf{D} | d^- \le d \le d^+\}$, where d^-, d^+ denote the beginning and the end of the interval, respectively (Markov, 1995 and Piasecka-Belkhayat, 2011). In the set of directed interval numbers two binary variables are defined. The first of them is the direction variable

$$\tau(\vec{d}) = \begin{cases} +, & \text{if } d^- \le d^+ \\ -, & \text{if } d^- > d^+ \end{cases} \tag{11}$$

and the other is the sign variable

$$\sigma(\vec{d}) = \begin{cases} +, & \text{if } d^- > 0, d^+ > 0 \\ -, & \text{if } d^- < 0, d^+ < 0 \end{cases}, \qquad \vec{d} \in \mathbf{D} \backslash \mathbf{Z} \tag{12}$$

where $\mathbf{Z} = \mathbf{Z_P} \cup \mathbf{Z_I} \in \mathbf{D}$ while $\mathbf{Z_P} = \{\vec{d} \in \mathbf{P} | d^- \le 0 \le d^+\}$, $\mathbf{Z_I} = \{\vec{d} \in \mathbf{I} | d^+ \le 0 \le d^-\}$ and **P** denotes a set of all directed proper intervals ($d^- \le d^+$), **I** denotes a set of all improper intervals ($d^- \ge d^+$).

This way the rules of the directed interval arithmetic are not the same as the rules of the classical interval arithmetic. This arithmetic is more complicated, but more useful because in the set of the directed interval numbers it is possible to obtain the number zero by subtraction of two identical intervals

$$\vec{d} - \vec{d} = \left[d^- - d^-, d^+ - d^+\right] = \bar{0} \tag{13}$$

and the number one as the result of the division of two identical intervals

$$\vec{d} / \vec{d} = [d^{-\sigma(\vec{d})} / d^{-\sigma(\vec{d})}, d^{\sigma(\vec{d})} / d^{\sigma(\vec{d})}] = \bar{0}, \qquad \vec{d} \in \mathbf{D} \backslash \mathbf{Z} \tag{14}$$

which was impossible when applying classical interval arithmetic (Popova, 2001).

3. Results of computations

As a numerical example the bio-heat transfer in a skin tissue of thickness L_3 = 12.1 mm has been analyzed. The following input data have been introduced: L_1 = 0.1 mm, L_2 = 2.1 mm, λ_1 = 0.235 W/(m·K), λ_2 = 0.445 W/(m·K), λ_3 = 0.185 W/(m·K), $c_1 = 4.3068 \cdot 10^6$ J/(m^3·K), $c_2 = 3.96 \cdot 10^6$ J/(m^3·K), $c_3 = 2.674 \cdot 10^6$ J/(m^3·K), $c_B = 3.9962 \cdot 10^6$ J/(m^3·K), $Q_{m1} = 0$, $Q_{m2} = Q_{m3}$ = 245 W/m^3, T_B = 37 °C, $G_{B1} = 0$, $G_{B2} = G_{B3}$ = 0.00125 (m$^3_{\text{blood}}$/s)/m$^3_{\text{tissue}}$, initial temperature $T_{10} = T_{20} = T_{30}$ = 37°C, the external heat source $q_b = 20 \cdot 10^3$ W/m^2 , the time step Δt = 0.001 s, the mesh step $\Delta x_e = (L_e - L_{e-1}) / n_e$ where $n_1 = 5$, $n_2 = 30$ and $n_3 = 60$. The time of external heat source exposition has been assumed as 5 s.

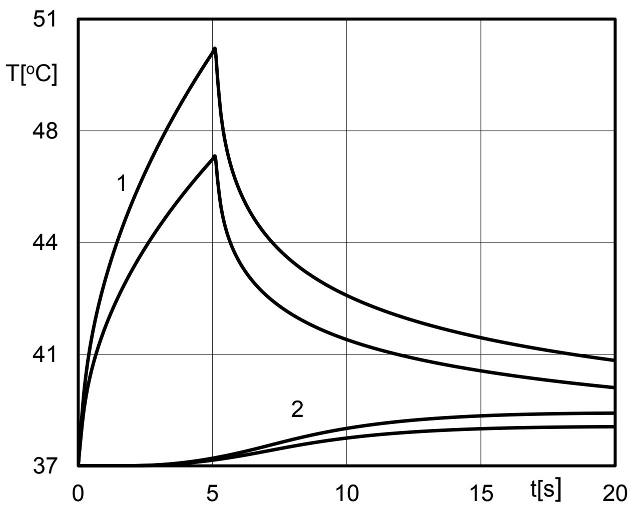

Fig. 1: Heating and cooling curves – classical interval arithmetic.

Fig. 1 presents the courses of the interval temperature functions at the selected internal nodes L_1 (1), L_2 (2) for interval values $\bar{c}_e = [c_e - 0.05\,c_e,\ c_e + 0.05\,c_e]$, $\bar{\lambda}_e = [\lambda_e - 0.05\,\lambda_e,\ \lambda_e + 0.05\,\lambda_e]$ and $\bar{Q}_{me} = [Q_{me} - 0.05\,Q_{me},\ Q_{me} + 0.05\,Q_{me}]$ applying the classical interval arithmetic.

Fig. 2 shows the courses of the interval temperature functions at the same nodes and for the same values of interval thermophysical parameters with the use of the rules of directed interval arithmetic.

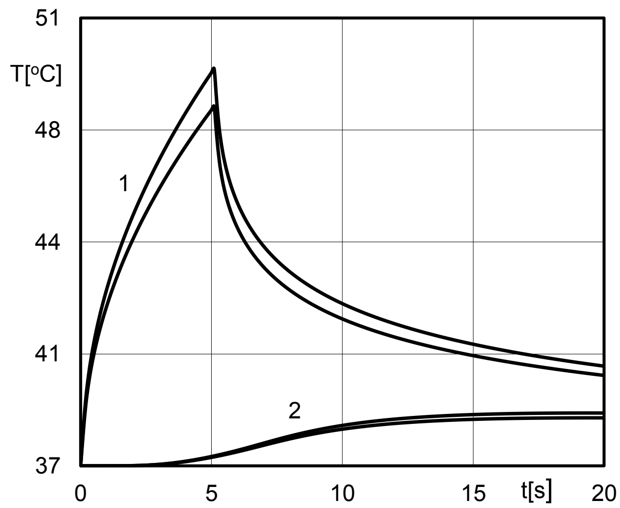

Fig. 2: Heating and cooling curves – directed interval arithmetic.

4. Conclusions

This paper presents the modelling of heat transfer in skin tissue using the interval finite difference method. The obtained temperatures were compared through applying directed and classical interval arithmetic. Using the rules of interval arithmetic allows to adapt imprecisely defined thermal parameters and due to this the temperatures are received as intervals. The temperatures calculated using the classical arithmetic are relatively wide and their width increases over time. On the other hand, the results attained with the directed arithmetic are much narrower and the difference between obtained temperatures does not rise over time.

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