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GLOBAL TOPOLOGY STIFFNESS OPTIMIZATION OF FRAME STRUCTURES BY MOMENT-SUM-OF-SQUARES HIERARCHY

Tyburec M.[*], Zeman J.[**], Kružík M.[***], Henrion D.[****]

Abstract: *This contribution develops an efficient formulation for the topology optimization of frame structures with fixed-aspect-ratio cross-sections, solvable to global optimality by the moment-sum-of-squares hierarchy. While the hierarchy generates a sequence of non-decreasing lower-bounds, we develop a sequence of feasible upper-bounds, allowing to assess the optimized design quality in each relaxation. Finally, these bounds provide a means of establishing a new sufficiency condition of global ε-optimality.*

Keywords: Moment-sum-of-squares hierarchy, Topology optimization, Frame structures, Global optimality.

1. Introduction

The design of frame structures constitutes one of the oldest applications in structural optimization. Assuming the ground structure approach (Dorn et al., 1964), in which the position of structural nodes and their connectivity are fixed in advance, two main branches can be distinguished: (i) optimizing discrete, or (ii) continuous cross-sectional properties of individual finite elements. In the discrete case, the problem exhibits a combinatorial nature, and thus the branch-and-bound method may be adopted to compute the guaranteed globally optimal solutions (Kanno, 2016), albeit with large computational expenses. For the continuous case a non-convex formulation is known only, and thus local optimization approaches are used. To the authors' knowledge, no method has been developed so far that allows for obtaining a guaranteed global optimum in the continuous setting.

In this contribution, we exploit the polynomial structure of the optimization problem, i.e., the objective function as well as the constraints are in fact low-degree polynomials and form a semi-algebraic set. Therefore, we can adopt the moment-sum-of-squares hierarchy (Lasserre, 2001) for the problem solution. Moreover, it will be shown that each relaxation of the moment-sum-of-squares hierarchy generates both lower- and upper-bound to the objective function, a measure of the actual design quality. Finally, based on these bounds we develop a new simple sufficiency condition of global ε-optimality.

2. Semidefinite programming formulation

Assuming a continuum design space discretized using n_e Euler-Bernoulli frame elements with given-shape fixed-aspect-ratio cross-sections, we search their sizes a to maximize the structural stiffness against specified loads $\mathbf{f}(\mathbf{a})$. The structural stiffness is measured (inversely) using the compliance c, work done by external forces. Then, the optimization problem can be posed as a non-linear (non-convex) semi-definite program

$$\min_{\mathbf{a},\mathbf{u}_j} \sum_{j=1}^{n_{\mathrm{lc}}} \omega_j \mathbf{f}_j(\mathbf{a})^{\mathrm{T}} \mathbf{u}_j \tag{1a}$$

[*] Ing. Marek Tyburec: Department of Mechanics, Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7/2077; 166 29, Prague; CZ, marek.tyburec@fsv.cvut.cz

[**] Prof. Ing. Jan Zeman, PhD.: Department of Mechanics, Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7/2077; 166 29, Prague; CZ, Jan.Zeman@cvut.cz

[***] Assoc Prof. RNDr. Martin Kružík, PhD.: Department of Physics, Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7/2077; 166 29, Prague; CZ, kruzik@utia.cas.cz

[****] Prof. Ing. Didier Henrion, PhD.: LAAS-CNRS, University of Toulouse, 7 Avenue du Colonel Roche, 31400 Toulouse; FR, henrion@laas.fr

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$$\text{s.t.} \begin{pmatrix} c & -\mathbf{f}(\mathbf{a})^{\mathrm{T}} \\ -\mathbf{f}(\mathbf{a}) & \mathbf{K}(\mathbf{a}) \end{pmatrix} \succcurlyeq \mathbf{0}, \tag{1b}$$

$$\boldsymbol{\ell}^{\mathrm{T}}\mathbf{a} \leq \overline{V}, \tag{1c}$$

$$\mathbf{a} \geq \mathbf{0}, \tag{1d}$$

in which $\mathbf{K}(\mathbf{a})$ denotes the structural stiffness matrix, $\boldsymbol{\ell}$ is the frame elements lengths column vector, $\overline{V}$ constitutes a prescribed volume upper bound, and the notation "$\succcurlyeq 0$" requires positive semidefiniteness of the left-hand-side matrix.

3. Polynomial optimization

3.1. Efficient formulation

In the optimization problem (1), the objective function (1a) and the constraints (1c) with (1d) are all polynomials of degree one. Moreover, the matrix inequality (1b) contains polynomial entries of the maximum degree two, and forms thus a semi-algebraic set. Consequently, polynomial optimization techniques are suitable for a solution of (1). In the following text we restrict ourselves to the moment-sum-of-squares hierarchy of Lasserre (2001).

Before proceeding, we reformulate the problem (1) slightly to improve the optimization performance. First, to satisfy the assumptions of the celebrated Putinar's Positivstellensatz (Putinar, P1993), we bound the design variables of (1) both from below and above. In the case of the cross-sectional areas, the lower bound is set already, and the upper bound can be established from the volume constraint (1c), i.e.,

$$0 \leq a_i \leq \frac{\overline{V}}{\ell_i}, \quad \forall i \in \{1, \dots, n_{\mathrm{e}}\}. \tag{2}$$

In the case of the compliance variable c, neither of the bounds is set explicitly yet. However, the zero lower bound comes from the definition of the compliance functional, and the upper bound is established by any feasible solution to (1). Indeed, one of these is the uniform distribution of the cross-sectional areas

$$0 \leq c \leq \mathbf{f}(\bar{\mathbf{a}})^{\mathrm{T}}\mathbf{K}(\bar{\mathbf{a}})^{-1}\mathbf{f}(\bar{\mathbf{a}}), \;\; where\; \bar{\mathbf{a}} = \frac{\overline{V}}{\mathbf{1}^{T}\boldsymbol{\ell}}\mathbf{1}. \tag{3}$$

Having constrained the bounds of the design variables, the design space is clearly bounded and closed, hence compact. We further rescale the domain of the design variables to $[-1,1]$, which considerably reduces numerical issues that may arise during the solution:

$$a_i = 0.5\overline{V}(a_{\mathrm{sc},i} + 1)/\ell_i, \;\; \forall i \in \{1, \dots, n_{\mathrm{e}}\}, \tag{4a}$$

$$c = 0.5\mathbf{f}(\bar{\mathbf{a}})^{\mathrm{T}}\mathbf{K}(\bar{\mathbf{a}})^{-1}\mathbf{f}(\bar{\mathbf{a}})(c_{\mathrm{sc}} + 1), \tag{4b}$$

where $\mathbf{a}_{\mathrm{sc}}$ and c_{sc} denote the scaled cross-sectional areas and compliance, respectively. Finally, the box constraints are replaced by the second-order polynomials

$$1 - a_{\mathrm{sc},i}^2 \geq 0, \;\; \forall i \in \{1, \dots, n_{\mathrm{e}}\}, \tag{5a}$$

$$1 - c_{\mathrm{sc}}^2 \geq 0, \tag{5b}$$

consequently reducing the number of constraints. Although this step might seem counter-intuitive, slightly complicating the problem structure as box constraints are easier to handle, the constraints (5) are tighter in the moment representation of the problem. To see this, assume that (y_0, y_1, y_2) are the moments associated with the canonical basis of the vector space of polynomials of the degree at most two, $(1, a_{\mathrm{sc},i}, a_{\mathrm{sc},i}^2)$ for some $i \in \{1, .., n_{\mathrm{e}}\}$. The same procedure can be applied, however, for all the variables $\mathbf{a}_{\mathrm{sc}}$ and c_{sc}. Then, in the first relaxation of the moment-sum-of-squares hierarchy, while Eq. (5a) becomes

$$y_0 - y_2 \geq 0 \tag{6}$$

the box constraint $-1 \leq a_{\mathrm{sc},i} \leq 1$ yields

$$-y_0 \leq y_1 \leq y_0, \tag{7}$$

with $y_0 = 1$. Moreover, the moment matrix of the entire optimization problem contains the principal submatrix

$$\begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix} \succcurlyeq 0 \tag{8}$$

that must be positive semi-definite as the entire moment matrix is.

For the quadratic constraint (5a), $y_2 \leq 1$ from Eq. (6) and $y_2 \geq 0$ because of Eq. (8). Writing the determinant of (8) then provides us with $y_0 y_2 \geq y_1^2$. Consequently, we observe that $0 \leq y_1^2 \leq y_2 \leq 1$, so that the first-order moment y_1 always satisfies (5a). Moreover, y_2 is upper-bounded by 1.

In the case of the box constraints, however, we only have $0 \leq y_1^2 \leq 1$, Eq. (7), and $y_1^2 \leq y_2$, Eq. (8). Note that there is no upper bound for y_2, which can attain arbitrarily large values in the first relaxation. From the mechanical point of view, this implies an arbitrarily-large rotational stiffness of the elements.

Combining these observations together, the final formulation reads

$$\min_{\mathbf{a}_{\mathrm{sc}}, c_{\mathrm{sc}}} \ 0.5\mathbf{f}(\bar{\mathbf{a}})^{\mathrm{T}}\mathbf{K}(\bar{\mathbf{a}})^{-1}\mathbf{f}(\bar{\mathbf{a}})(c_{\mathrm{sc}}+1), \tag{9a}$$

$$\text{s.t.} \ \begin{pmatrix} 0.5\mathbf{f}(\bar{\mathbf{a}})^{\mathrm{T}}\mathbf{K}(\bar{\mathbf{a}})^{-1}\mathbf{f}(\bar{\mathbf{a}})(c_{\mathrm{sc}}+1) & -\mathbf{f}(\mathbf{a}_{\mathrm{sc}})^{\mathrm{T}} \\ -\mathbf{f}(\mathbf{a}_{\mathrm{sc}}) & \mathbf{K}(\mathbf{a}_{\mathrm{sc}}) \end{pmatrix} \succcurlyeq \mathbf{0}, \tag{9b}$$

$$2 - n_{\mathrm{e}} - \mathbf{1}^{\mathrm{T}}\mathbf{a}_{\mathrm{sc}} \geq 0, \tag{9c}$$

$$1 - a_{\mathrm{sc},i}^2 \geq 0, \ \ \forall i \in \{1, \dots, n_{\mathrm{e}}\}, \tag{9d}$$

$$1 - c_{\mathrm{sc}}^2 \geq 0. \tag{9e}$$

3.2. Solution process

Let $\mathbf{x} = (\mathbf{a}_{\mathrm{sc}}, c_{\mathrm{sc}})$ be the vector of the design variables, and let $p_j, j \in \{0, \dots, n_{\mathrm{e}}+2\}$, denote the polynomials in (9a), (9c), (9d) and (9e). Moreover, let $P_{n_{\mathrm{e}}+3}$ stand for the polynomial matrix inequality (9b). The canonical basis associated with the vector space of polynomials of the degree at most d reads as

$$\mathbf{b}_d(\mathbf{x}) = \left(1, x_1, x_2, \dots, x_{n_{\mathrm{e}}+1}, x_1^2, x_1x_2, \dots, x_1x_{n_{\mathrm{e}}+1}, x_2^2, x_2x_3, x_{n_{\mathrm{e}}+1}^2, \dots, x_1^d, \dots, x_{n_{\mathrm{e}}+1}^d\right). \tag{10}$$

Then, each of the polynomials p_j can be expressed as a linear combination of the monomials,

$$p_j(\mathbf{x}) = \sum_{\beta=1}^{|\mathbf{b}_d(\mathbf{x})|} p_{\mathrm{c},j,\beta}\mathbf{x}^{\boldsymbol{\alpha}_\beta}, \ \ \text{where } \mathbf{x}^{\boldsymbol{\alpha}_\beta} = \prod_{m=1}^{n_{\mathrm{e}}+1} x_m^{\alpha_{\beta,m}}, \tag{11}$$

which are associated with the polynomial vector space basis $\mathbf{b}_d(\mathbf{x})$. In Eq. (11), $p_{\mathrm{c},j,\beta}$ is the coefficient of the linear combination associated with the *j*-th polynomial p_j and with the β entry in $\boldsymbol{b}_d(\mathbf{x})$, and $\boldsymbol{\alpha}_\beta$ is the multi-index vector whose entries $\boldsymbol{\alpha}_{\beta,m}, m \in \{1, \dots, n_{\mathrm{e}}+1\}$, are non-negative integers associated with the entries in $\mathbf{b}_d(\mathbf{x})$. Therefore, $\sum_{m=1}^{n_{\mathrm{e}}+1} \boldsymbol{\alpha}_{\beta,m} \leq d$.

In addition, let $\mathbf{y}$ denote the moments corresponding to the monomials in the basis $\mathbf{b}_d(\mathbf{x})$. Then, we build the moment-sum-of-squares hierarchy of convex outer approximations as

$$\min_{\mathbf{y}} \ \sum_{\beta=1}^{|\mathbf{b}_d(\mathbf{x})|} p_{\mathrm{c},0,\beta} y_\beta, \tag{12a}$$

$$\text{s.t.} \qquad \mathbf{M}_d(\mathbf{y}) \succcurlyeq 0, \tag{12b}$$

$$\mathbf{M}_{d-v_j}(\mathbf{y}) \succcurlyeq 0, \ \ \forall j \in \{1, \dots, n_{\mathrm{e}}+3\}, \tag{12c}$$

which are solved successively with an increasing relaxation order $d \in \mathbb{N}$. In (12), the matrix $\mathbf{M}_d(\mathbf{y})$ is the moment matrix of the *d*-th order, and $\mathbf{M}_{d-v_j}$ is the $(d-v_j)$-th order localizing matrix associated with p_j or P_j, and v_j is one half of the degree of p_j or P_j. For more details about these matrices, we refer the reader to (Lasserre, 2001; Henrion and Lasserre, 2006).

The hierarchy of convex linear semidefinite programs (12) possesses a monotonous convergence, making the outer approximations tighter with increasing d and providing non-decreasing lower-bounds on the globally optimal solution.

3.3. Bounds on the global optimum

Let $\mathbf{y}_d^*$ denote the optimal solution of the d-th order relaxation of (12). Then, the objective function value is a lower-bound to the globally optimal compliance c^*,

$$\underline{c} \coloneqq 0.5\mathbf{f}(\bar{\mathbf{a}})^{\mathrm{T}}\mathbf{K}(\bar{\mathbf{a}})^{-1}\mathbf{f}(\bar{\mathbf{a}})\left(y_{\mathrm{c}}^{*(1)}+1\right) \leq c^*, \tag{13}$$

where $y_{\mathrm{c}}^{*(1)}$ stands for the first-order moment associated with c_{sc} in $\mathbf{y}_d^*$. In addition, let $\mathbf{y}_{\mathrm{a}}^{*(1)}$ denote the first-order moments associated with $\mathbf{a}_{\mathrm{sc}}$ extracted from $\mathbf{y}_d^*$. Putting $(\mathbf{a}_{\mathrm{sc}}, c_{\mathrm{sc}}) = \left(\mathbf{y}_{\mathrm{a}}^{*(1)}, y_{\mathrm{c}}^{*(1)}\right)$ into (9), Eq. (9c) is clearly satisfied, together with (9d) and (9e) as is shown in Section 3.1. On the contrary, the non-convex constraint (9b) may be violated. To satisfy (9b) it is sufficient to increase the compliance, so that the structural equilibrium holds. Indeed, using the (generalized) Schur complement lemma, e.g., (Gallier, 2011, Theorem 16.1), on (9d), we have

$$\overline{c} = \mathbf{f}\left(\mathbf{y}_{\mathrm{a}}^{*(1)}\right)^{\mathrm{T}}\mathbf{K}\left(\mathbf{y}_{\mathrm{a}}^{*(1)}\right)^{\dagger}\mathbf{f}(\mathbf{y}_{\mathrm{a}}^{*(1)}), \tag{14}$$

with $(\bullet)^{\dagger}$ denoting the Moore-Penrose pseudo-inverse of •, which makes the problem (9) feasible while increasing the objective function to the upper-bound value $\overline{c}$. Note that (14) basically evaluates the structural compliance using the finite-element analysis.

Consequently, we have $\underline{c} \leq c^* \leq \overline{c}$ in each of the relaxation of the hierarchy (12). Moreover, as $\overline{c} - \underline{c} \to 0$ we approach the global optimum and

$$\mathbf{f}\left(\mathbf{y}_{\mathrm{a}}^{*(1)}\right)^{\mathrm{T}}\mathbf{K}\left(\mathbf{y}_{\mathrm{a}}^{*(1)}\right)^{\dagger}\mathbf{f}\left(\mathbf{y}_{\mathrm{a}}^{*(1)}\right) - 0.5\mathbf{f}(\bar{\mathbf{a}})^{\mathrm{T}}\mathbf{K}(\bar{\mathbf{a}})^{-1}\mathbf{f}(\bar{\mathbf{a}})(y_{\mathrm{c}}^{*(1)}+1) < \varepsilon \tag{15}$$

is a sufficient condition of global ε-optimality. Notice that this optimality condition is much easier to check than the traditional one (Curto and Fialkow, 2000) which relies on the more expensive rank computation.

4. Conclusions

This contribution develops an efficient formulation for topology optimization of frame structures with fixed aspect-ratio cross-sections. Because the feasible space of the formulation forms a semi-algebraic set, the problem admits a solution by the moment-sum-of-squares hierarchy. Moreover, while the hierarchy generates a sequence of non-decreasing lower bounds, a sequence of upper bounds can be established by computing the structural compliance with the cross-sectional areas defined by the corresponding first-order moments. Using these bounds, a simple sufficient condition of ε-optimality is established, eliminating the need for the traditional rank-based computations.

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