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APPLICATION OF FIRST INTEGRALS IN THE CONSTRUCTION OF THE LYAPUNOV FUNCTION FOR THE RANDOM RESPONSE STABILITY TESTING

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Abstract: *The paper deals with a possibility of using the properties of first integrals for the construction of Lyapunov function for the analysis of a dynamic system stability in the stochastic domain. It points out certain characteristics of first integrals resulting in the necessity to introduce additional constraints to assure the principal properties of the Lyapunov function. A number of these constraints has their physical interpretation with reference to system stability. The advantage of this method constructing the Lyapunov function consists in the fact that the Lyapunov function itself contains information on the examined system and, consequently, it is not merely a positive definite function without any relation to the actual case concerned. The presented theory finds application in many dynamical systems. The procedure is illustrated by a nonlinear SDOF example.*

Keywords: Stochastic stability, Lyapunov function, First integrals, Cyclic coordinates.

1. Introduction

Dynamic stability of systems in which random components of system parameters or excitations cannot be neglected, represent a natural extension of exclusively deterministic cases. Among the various methods applicable for the stability analysis of such systems, the important position belongs to the second Lyapunov method extended for cases with random excitation. Especially in the stochastic domain is this method considered to provide a better understanding of the overall properties of the system structure in terms of stability of a given type. It also helps with an analyzis of the broader context of system parameters and their random perturbations. However, if it is possible to formulate the necessary and sufficient conditions for the existence of the Lyapunov function (LF) for the appropriate type of stability of the system, LF can serve as a powerful tool not only for theoretical analysis, but also for direct practical applications. This particular research is motivated by the concept of the structural health monitoring of a bridge by indirect measurement during the crossing of a special measuring vehicle, (Bayer and Urushadze, 2021), because the data obtained from such measurements usually contain a strong random component.

Probably the first use of Lyapunov function adapted to stochastic stability problems was published by Bertram and Sarachik (1959). In subsequent studies by Tikhonov and Mironov (1977) or Bolotin (1984) is the total time derivative of a positive definite function, which is considered as Lyapunov function in deterministic problems, replaced in the stochastic domain by the adjoint Fokker-Planck (FP) operator:

$$\mathbf{L}\{\lambda(t,\mathbf{u})\} \;=\; \frac{\partial\lambda(t,\mathbf{u})}{\partial t} + \sum_{i=1}^{n}\frac{\partial\lambda(t,\mathbf{u})}{\partial u_i}\kappa_i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\lambda(t,\mathbf{u})}{\partial u_i\partial u_j}\kappa_{ij}, \tag{1}$$

where κ_i, κ_{ij} are the drift and diffusion coefficients of the n-dimensional Markov process, m depends on the system structure:

$$\kappa_i = \sum_{k=1}^{m} A_{ik}(t)\cdot f_{ik}(\mathbf{u}) + \frac{1}{2}\sum_{k,l=1}^{m}\sum_{p=1}^{n}\frac{\partial f_{ik}(\mathbf{u})}{\partial u_p} f_{ip}(\mathbf{u})\cdot s_{iklp}\,, \qquad \kappa_{ij} = \sum_{k,l=1}^{m} f_{ik}(\mathbf{u}) f_{jl}(\mathbf{u})\cdot s_{ikjl}\,. \tag{2}$$

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Equations (1,2) relate to the original stochastic system, the stochastic stability of which is assessed:

$$\dot{u}_i = \sum_{k-1}^{m} (A_{ik}(t) + w_{ik}(t)) f_{ik}(\mathbf{u}); \quad \mathbf{u}(t_0) = \mathbf{u}_0. \tag{3}$$

where $\lambda(t, \mathbf{u})$—candidate for the LF, $A_{ik}(t)$—nominal values of system coefficients, $w_{ik}(t)$—Gaussian white noise of cross-intensity s_{ikjl}, and $f_{ik}(\mathbf{u})$—continuous non-decreasing functions.

Let $\lambda(t, \mathbf{u})$ be positive definite and continuous together with its first derivative with respect to time and the second derivative with respect to space. If $\psi(t, \mathbf{u}) = \mathbf{L}\{\lambda(t, \mathbf{u})\}$ is negative in the relevant domain $\boldsymbol{\Omega}$ and vanish or does not exist in the origin, $\lambda(t, \mathbf{u})$ can be considered as the Lyapunov function. Obviously, for any $||\mathbf{u}_0|| \neq 0$ function $\lambda(t, \mathbf{u})$ decreases for $t \to \infty$ and approaches zero simultaneously in all spatial coordinates $\mathbf{u}$. This means that the trivial solution of the system Eq. (3) is stable in terms of probability.

It turns out that the Lyapunov function can be constructed on the basis of the first integrals of the corresponding deterministic system. The simplest example of this approach is the use of the overall energy balance, which is also the most common first integral of the system. Such a procedure is presented by Chetayev (1962) for the case of assessing stability in the deterministic area. This paper attempts to use the first integrals to construct a Lyapunov function that could be used to stochastically evaluate the stability of one class of systems with polynomial nonlinearities and Gaussian parametric and additive white noise.

2. Single-Degree-of-Freedom example system

For the sake of simplicity, an example motivated by an single-degree-of-freedom aeroelastic model will be presented here, on which it will be possible to clearly show the construction of the LF. In this model, the movement of a prismatic body perpendicularly to an air flow results from an aeroelastic interaction of a streaming medium and a moving body, e.g., (Novák and Davenport, 1970),

$$M \cdot \ddot{u}(t) + F_{dam}(u, \dot{u}) + C \cdot u(t) = 0\,, \tag{4}$$

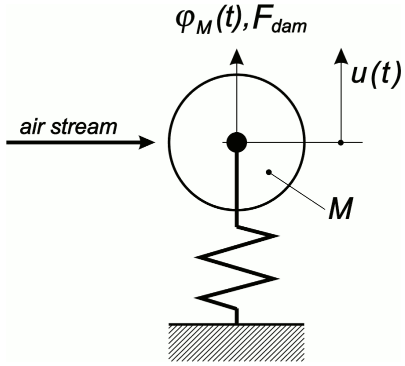

Fig. 1: Outline of an SDOF system subdued to a cross air stream.

where M, C are the mass and stiffness matrices, respectively, and $F_{dam}(u, \dot{u})$ denotes the nonlinear damping force. Excitation enters the equation as an action of the non-linear damping. A change in the lifting force due to the variation in the angle of attack can cause self-excitation. Random air pressure fluctuations have the same character and can be considered as a perturbation of the damping coefficient. The most frequently used choices of damping result in the Rayleigh or the van der Pol type equation:

(i) The Rayleigh type $F_{dam}(u, \dot{u}) = 2M \left(\omega_b + \frac{1}{2}\delta^2 \dot{u}^2(t)\right) \dot{u}(t)$:

$$\begin{aligned} \dot{u_1} &= u_2\,; \\ \dot{u_2} &= (2\omega_b - \delta^2 u_2^2 + w_2) \cdot u_2 - (\omega_0^2 + w_1) \cdot u_1 \end{aligned} \tag{5}$$

(ii) The van der Pol type $F_{dam}(u, \dot{u}) = 2M \left(\omega_b + \frac{1}{2}\gamma^2 u^2(t)\right) \dot{u}(t)$:

$$\begin{aligned} \dot{u_1} &= u_2\,; \\ \dot{u_2} &= (2\omega_b - \gamma^2 u_1^2 + w_2) \cdot u_2 - (\omega_0^2 + w_1) \cdot u_1 \end{aligned} \tag{6}$$

where the phase space $\mathbf{u} = \{u_1 = u, u_2 = \dot{u}\}$ has been introduced. The linear and nonlinear damping coefficients ω_b, γ, δ comprise structural damping and aeroleastic self-exciting effects. Parametric noises w_1, w_2 with cross-intensities $s_{k,l}$ affect eigenfrequency and damping, respectively. The systems (5,6) have the first integrals of the total energy type and the corresponding Lyapunov functions can have the form:

(i) $$\lambda(t, \mathbf{u}) = \frac{1}{2}u_2^2 + \frac{1}{2}\omega_0^2 u_1^2 \tag{7}$$

(ii) $$\lambda(t, \mathbf{u}) = \frac{1}{2}(u_2 - G(u_1))^2 + \frac{1}{2}\omega_0^2 u_1^2 \tag{8}$$

where $$G(u_1) = \int_0^{u_1} \left(2\omega_b - \gamma^2 \zeta^2\right) d\zeta = 2\omega_b u_1 - \frac{1}{3}\gamma^2 u_1^3\,; \tag{9}$$

Equations (7), (8) represent positive definite functions. With respect to Eqs. (2) and (3), the diffusion coefficients κ_i, κ_{ij} and the derivatives of the function $\lambda(t, \mathbf{u})$ according to phase variables have the form of:

$$\text{(i)} \qquad \begin{cases} \kappa_1 = u_2\,, & \kappa_2 = (2\omega_b - \delta^2 \cdot u_2^2) \cdot u_2 - \omega_0^2 \cdot u_1 \\ \kappa_{11} = \kappa_{12} = \kappa_{21} = 0\,, & \kappa_{22} = u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22} \\ \partial_{u_1} \lambda(t, \mathbf{u}) = \omega_0^2 u_1\,, & \partial_{u_2} \lambda(t, \mathbf{u}) = u_2\,, \qquad \partial_{u_2^2} \lambda(t, \mathbf{u}) = 1 \end{cases} \tag{10}$$

$$\text{(ii)} \qquad \begin{cases} \kappa_1 = u_2\,, & \kappa_2 = (2\omega_b - \gamma^2 \cdot u_1^2) \cdot u_2 - \omega_0^2 \cdot u_1 \\ \kappa_{11} = \kappa_{12} = \kappa_{21} = 0\,, & \kappa_{22} = u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22} \\ \partial_{u_1} \lambda(t, \mathbf{u}) = (u_2 + 2\omega_b u_1 - 1/3\,\gamma^2 u_1^3)(2\omega_b - \gamma^2 u_1^2) + \omega_0^2 u_1\,, \\ \partial_{u_2} \lambda(t, \mathbf{u}) = (u_2 + 2\omega_b u_1 - 1/3\,\gamma^2 u_1^3)\,, \qquad \partial_{u_2^2} \lambda(t, \mathbf{u}) = 1 \end{cases} \tag{11}$$

where $n = 2, m = 3$, $f_{12} = f_{22} = u_2$, $f_{21} = u_1$, $f_{23} = u_2^3$ or $u_1^2 u_2$, $w_{21} = -w_1, w_{22} = w_2$; $s_{ikjl} = s_{kl}$ for $i = 2$ or $j = 2$, and $s_{ikjl} = 0$ when any from the following holds: $i = 1$, $j = 1$, $k = 3$ or $l = 3$.

A summation of these partial expressions yields: $\mathbf{L}\{\lambda(t, \mathbf{u})\}$:

$$\text{(i)} \qquad \mathbf{L}\{\lambda(t, \mathbf{u})\} = \psi(t, \mathbf{u}) = u_2^2 (2\omega_b - \delta^2 u_2^2) + u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22} \tag{12}$$

$$\text{(ii)} \qquad \psi(t, \mathbf{u}) = \omega_0^2 u_1^2 (2\omega_b - \frac{1}{3}\gamma^2 u_1^2) + (u_1^2 s_{11} + u_1 u_2 (s_{12} + s_{21}) + u_2^2 s_{22}) \tag{13}$$

Let us deal with the case, when the noises w_1, w_2 are independent, i.e. when $s_{12} = s_{21} = 0$. The basic structure of Eqs (12), (13) reveals that in comparison to the deterministic case, both noises have destabilizing effects. Equation $\psi(t, \mathbf{u}) = 0$ can be used for the estimation of the boundaries of stability of the initial system. The system comprises only symmetrical independent parametric noises and no other excitations, as a result of which its response will be symmetrical, too. Thus, processes u_1, u_2 can be assumed centered.

If the process u_1 can be considered at least approximately a Gaussian, $D_{11}^4 = 3(D_{11}^2)^2$ where D_{11}^2 and D_{11}^4 are central second and fourth moments of u_1, respectively. This approximation is probably permissible with regard to the fact that the characteristic of the systems Eqs (5), (6) is linear. In such a case, see (Náprstek and Fischer, 1999), it can be shown that the estimated stability boundary for $D_{11}^2 > 0, D_{22}^2$ has the form of

$$\text{(i)} \qquad -\frac{3\delta^2}{s_{11}} (D_{22}^2)^2 + \frac{(2\omega_b + s_{22})}{s_{11}} D_{22}^2 + D_{11}^2 = 0 \tag{14}$$

$$\text{(ii)} \qquad -\frac{\omega_0^2 \gamma^2}{s_{22}} \cdot (D_{11}^2)^2 + \frac{(2\omega_0^2 \omega_b + s_{11})}{s_{22}} D_{11}^2 + D_{22}^2 = 0 \tag{15}$$

The curves Eqs (14), (15) are parabolas passing through the origin, the direction of their axes is (i) horizontal and (ii) vertical, respectively. The stable state results from the negative value of the function $\psi(t, \mathbf{u})$, i.e. the area between the respective parabola and (i) positive vertical axis of D_{22}, and (ii) positive horizontal axis of D_{11}^2 respectively. This can give rise to three different cases. If the apex of the parabola is below the axis D_{11}^2 or to the left of the axis D_{22}^2, the stability area extends to the origin. That holds, if:

$$\text{(i)} \quad 2\omega_b + s_{22} < 0\,; \qquad \text{(ii)} \quad 2\omega_0^2 \omega_b + s_{11} < 0 \tag{16}$$

If the apex of the parabola coincides with the origin, i.e. the inequalities Eqs (16) become equations, the system is stable in a neighborhood of the origin. The negative or at least zero values of $\psi(t, \mathbf{u})$ can be attained without any of the quantities D_{11}^2, D_{22}^2 having to be positive.

If any of the expressions Eqs (16) is positive, the apex of the respective parabola is above the axis of D_{11}^2 or to the right of the axis of D_{22}^2. The system will acquire secondary stability due to the nonlinear term and will vibrate in a certain band of the width different from zero. The domain of stability adjoins to the vertical axis of D_{22}^2 or to the horizontal axis of D_{11}^2 beginning with the point

$$\text{(i)} \quad D_{22}^2 = \frac{2\omega_b + s_{22}}{3\delta^2}\,; \qquad \text{(ii)} \quad D_{11}^2 > \frac{2\omega_0^2 \omega_b + s_{11}}{\omega_0^2 \gamma^2} \tag{17}$$

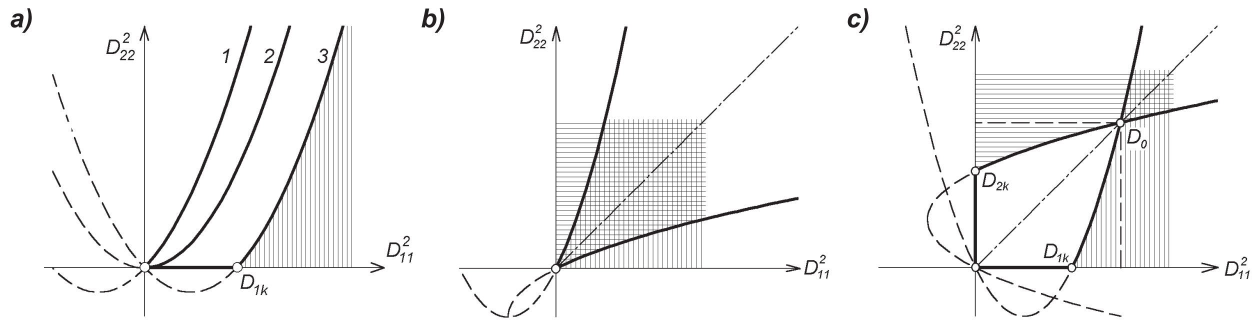

Fig. 2: (a) Stability domains of the van der Pol system (flow speed:1 — sub-critical, 2 — critical, 3 — super-critical);(b,c) Comparison of stability domains of van der Pol (parabola with vertical axis) and Rayleigh (parabola with horizontal axis) systems for sub-critical (b) or super-critical (c) flow speed, respectively.

If the air flow velocity or the value of ω_b is so low that the stability areas in both cases extend to the origin, the system is stable and the perturbations of displacements due to parametric noises do not differ practically. The behavior of the system is not visibly influenced by the selection of the damping model.

In supercritical regime an instability domain common to both models arises. The response structure in these cases depends considerably on the damping model. With increasing air flow velocity, however, we move more or less along the first quadrant diagonal. Starting with a certain velocity we arrive again at the stability domain determined by nonlinear part in the damping force. This domain is common to both models characterized once again by the dropping dependence on the type of nonlinear damping force model. Similar tendencies can be observed also for TDOF models, see, e.g., (Nabergoj and Tondl, 1994).

3. Conclusion

It is coming to light that the Lyapunov function constructed on the basis of first integrals has better properties than that constructed by other methods. In other words, if such LF can be constructed, it allows its user to obtain a broad overview of the nature of the system in terms of its global and local stability. For linear systems, the application of such LF yields results corresponding with the application of Routh-Hurwitz criteria and represents, consequently, the necessary and sufficient conditions. In nonlinear cases it is possible to base the investigation on the properties of the deterministic system, as the influence of parametric noises can be separated distinctly. This type of analysis has been shown using a simple SDOF example. The presented results showed a detailed insight into the stability properties of the investigated problem. The extension of the approach in the case of moving loads of the beam by periodic force and the stability of the respective response is straightforward. However, subsequent analysis of the stochastic properties of the measured data to study the properties of the supporting structure will require further effort.

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