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IMPLEMENTATION OF A HYBRID TREFFTZ FINITE ELEMENT FOR ELASTODYNAMIC MEDIA

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Abstract: *The paper develops numerical tools for dynamic analysis of elastic media. Particularly, the hybrid Trefftz method is applied in order to approximate the solution of the underlying differential equation expressed in the frequency domain. The main objective of this work is to implement the hybrid Trefftz method for numerical analysis of 2D elastodynamic media.* MATLAB *software is used as the programming language for the code development. To validate the implemented method, the results are compared to the analytical solutions.*

Keywords: Hybrid Trefftz method, Elastodynamics, Finite element analysis.

1. Introduction

The problem of dynamically loaded media is frequently investigated, especially in the fields of civil engineering and geotechnics. Under certain simplifications, such physical behaviour can be described by a set of coupled partial differential equations expressed in terms of an unknown displacement field depending on the time and space coordinates. For most of the practical cases this mathematical problem cannot be solved analytically and therefore numerical methods need to be applied to approximate the solution. By transferring all the field equations into the frequency domain, the original problem in time and space is divided into a number of subproblems, which are however formulated in terms of space coordinates only. The associated space solution is subsequently approximated. Various methods, such as FEM, were developed to tackle such task, however, for higher excitation frequencies a fine domain discretization is required, which results in computationally expensive simulation. Alternative options for such analysis are Trefftz methods, in which the unknown field is approximated using special shape functions which are required to satisfy the governing differential equations. In the paper the hybrid Trefftz method is investigated, in which the boundary traction field is additionally approximated on the boundary of the individual elements. The work of (Freitas, 1997) and (Moldovan, 2008) motivated to investigate the main characteristics of this method and its differences to other numerical strategies by applying it to an elastodynamic structure.

2. Problem Description

The behaviour of a loaded body represented by domain V with boundary Γ can be described using three main sets of equations, which are equilibrium equations (1), kinematic equations (2) and material law (3):

$$\boldsymbol{\mathcal{D}}\boldsymbol{\sigma} + \boldsymbol{b} = \rho\ddot{\boldsymbol{u}} \text{ in } V, \tag{1}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{D}}^T\boldsymbol{u} \text{ in } V, \tag{2}$$

$$\boldsymbol{\sigma} = \boldsymbol{k}\boldsymbol{\varepsilon} \text{ in } V. \tag{3}$$

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Vector $\boldsymbol{u}$ collects individual displacement components, vector $\boldsymbol{b}$ contains body forces and ρ denotes the mass density. Vectors $\boldsymbol{\sigma}$ and ε consist of components of the stress and strain tensors respectively and matrix $\boldsymbol{\mathcal{D}}$ is a differential operator matrix. As the material is considered as linear elastic and isotropic, the material matrix $\boldsymbol{k}$ is constant and two Lamé coefficients λ and μ are sufficient for its definition. To pose a valid problem, displacement and traction boundary conditions are introduced:

$$\boldsymbol{u} = \boldsymbol{u}_{\boldsymbol{\Gamma}} \text{ on } \Gamma_u \,, \; \boldsymbol{t} = \boldsymbol{t}_{\boldsymbol{\Gamma}} \text{ on } \Gamma_\sigma, \tag{4}$$

$\boldsymbol{u}_{\boldsymbol{\Gamma}}$ and $\boldsymbol{t}_{\boldsymbol{\Gamma}}$ denote the vectors of prescribed displacement and traction components. Symbols Γ_u and Γ_σ represent the Dirichlet and Neumann parts of the boundary Γ. The boundary traction vector is calculated based on the boundary equilibrium $\boldsymbol{t} = \boldsymbol{N}\boldsymbol{\sigma}$, where matrix $\boldsymbol{N}$ collects components of the outward unit normal. In the scope of this paper the loading is assumed to be a periodic function in time, therefore only the periodic solution is of interest and no information regarding the initial state of the structure needs to be provided. Combining the previously mentioned equations (1) to (3), the governing differential equation

$$\boldsymbol{\mathcal{D}} \boldsymbol{k} \boldsymbol{\mathcal{D}}^T \boldsymbol{u} - \rho \ddot{\boldsymbol{u}} + \boldsymbol{b} = \boldsymbol{0} \text{ in } V \tag{5}$$

is formed. It expresses a system of second order partial differential equations in time and space and is referred to as the Lamé equation.

In this work the frequency domain analysis method is applied in order to simplify the solution procedure of the previously derived equation. With the help of the Fourier series expansion, all the mentioned fields and equations can be transformed into the frequency domain and hence the problem depending on space and time coordinates is transferred into a number of sub problems which depend on the space coordinates only. The spectral form of the Lamé equation then reads

$$\boldsymbol{\mathcal{D}} \boldsymbol{k} \boldsymbol{\mathcal{D}}^T \boldsymbol{u_k} + \omega_k^2 \rho \boldsymbol{u_k} + \boldsymbol{b_k} = \boldsymbol{0}. \tag{6}$$

Vector $\boldsymbol{u_k}$ denotes the space part of the displacement vector obtained for harmonically oscillating loading with circular frequency ω_k and amplitude $\boldsymbol{b_k}$. The complete solution $\boldsymbol{u}$ is then recovered by superposition of the individual solutions for all considered circular frequencies ω_k. The solution procedure of eq. (6), and hence the acquisition of the space component $\boldsymbol{u_k}$ for a single circular frequency ω_k, is the main objective of this work and will be described in the next section.

3. Hybrid Trefftz Method

Similarly to FEM, also Trefftz methods discretize the domain into a number of finite elements, where a certain field is approximated by shape functions multiplied by unknown coefficients. In the case of Trefftz methods, the basis functions are restricted to satisfy the homogeneous part of the governing differential equation. Regarding the hybrid Trefftz method, the individual functions may violate the prescribed boundary conditions, hence a finite number of such basis functions is combined so that the boundary values of the resulting function get closer to the prescribed boundary conditions. The adjective hybrid indicates that more than one field is approximated simultaneously and independently. In the scope of this paper, the displacement field is approximated in the domain and the boundary traction field is approximated on the Dirichlet boundary. The purpose of the boundary field approximation is to enforce the boundary conditions and the continuity between adjacent elements. For simplicity, 2D plane strain situation is assumed for the upcoming derivations.

The displacement field $\boldsymbol{u_k} \approx \boldsymbol{U}\boldsymbol{X} + \boldsymbol{u_0}$ is approximated inside the element domain V^e. The displacement basis collected in matrix $\boldsymbol{U}$ is restricted to satisfy the homogeneous part of the spectral Lamé equation (6) and vector $\boldsymbol{u_0}$ is constructed as the particular solution of eq. (6). The strain field $\boldsymbol{\varepsilon_k} \approx \boldsymbol{\mathcal{D}}^T \boldsymbol{U}\boldsymbol{X} + \boldsymbol{\mathcal{D}}^T \boldsymbol{u_0} = \boldsymbol{E}\boldsymbol{X} + \boldsymbol{\varepsilon_0}$ is restricted to directly satisfy the kinematic equations, therefore $\boldsymbol{E}$ is not an independent basis. To construct the displacement and strain bases $\boldsymbol{U}$ and $\boldsymbol{E}$, it is necessary to solve the homogeneous part of the spectral Lamé equation. By the application of the Helmholtz decomposition

$$\boldsymbol{u_k} = \boldsymbol{\nabla} \Phi_p + \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla} \Phi_s, \tag{7}$$

with $\boldsymbol{\nabla}$ and $\boldsymbol{\epsilon}$ being gradient and Levi-Civita symbol, eq. (6) can be reformulated in terms of unknown irrotational potential Φ_p and a solenoidal potential Φ_s and hence decoupled into two independent Helmholtz equations. Their solution is subsequently sought in form

$$\Phi_{\alpha,n} = W_n(k_\alpha r) \exp(\mathrm{i} n \theta), \tag{8}$$

where r and θ are polar coordinates, n is an integer, $\alpha \in \{p, s\}$ and k_α stands for the wavenumber. It can be shown that function W_n must be chosen as a solution of the Bessel equation, which can be the Bessel function of the first or second kind or Hankel function of the first or second kind. For each order n, two basis functions can be derived combining eqs. (7) and (8), one associated to the dialatational part $\boldsymbol{\nabla}\Phi_{p,n}$ and the other to the shear part $\boldsymbol{\epsilon} \cdot \boldsymbol{\nabla}\Phi_{s,n}$. To construct the approximation matrix $\boldsymbol{U}$, orders $-N < n < N$ are considered, therefore the basis contains $2(2N+1)$ terms with N being the chosen maximum order.

The second field to be approximated contains the tractions $\boldsymbol{t_k} \approx \boldsymbol{Zp}$ on the Dirichlet element boundary Γ_u^e, which contains not only the external part of boundary, where the displacements are prescribed, but also the inter-element edges, where displacement continuity needs to be enforced. Matrix $\boldsymbol{Z}$ collects the boundary approximation basis and $\boldsymbol{p}$ stands for the vector of unknown coefficients. In this work the Chebyshev polynomials of type I up to maximum order M are adopted for the basis $\boldsymbol{Z}$. As the individual traction components are approximated independently, the matrix $\boldsymbol{Z}$ consists of $2(M+1)$ terms.

The finite element system of equations is derived using the weighted residual method. Firstly, the equilibrium equations are weakly imposed while the displacement approximation basis is used as the weighting matrix. On the other hand, the kinematic equations, the material law and the traction boundary condition are fulfilled locally. Combining and manipulating the previously mentioned equations, the first matrix equation of the complete system of algebraic equations

$$\begin{bmatrix} \boldsymbol{D} & -\boldsymbol{B} \\ -\widehat{\boldsymbol{B}}^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{t_\Gamma}} - \overline{\boldsymbol{t_{\Gamma_0}}} \\ \overline{\boldsymbol{u_{\Gamma_0}}} - \overline{\boldsymbol{u_\Gamma}} \end{bmatrix} \tag{9}$$

is derived. The definitions of the individual matrices and vectors read

$$\begin{aligned} \boldsymbol{D} &= \int_{\Gamma^e} \widehat{\boldsymbol{U}}^T \boldsymbol{NkE}\,\mathrm{d}\Gamma,\ \boldsymbol{B} = \int_{\Gamma_u^e} \widehat{\boldsymbol{U}}^T \boldsymbol{Z}\,\mathrm{d}\Gamma,\ \overline{\boldsymbol{t_\Gamma}} = \int_{\Gamma_\sigma^e} \widehat{\boldsymbol{U}}^T \boldsymbol{t_\Gamma}\,\mathrm{d}\Gamma, \\ \overline{\boldsymbol{t_{\Gamma_0}}} &= \int_{\Gamma^e} \widehat{\boldsymbol{U}}^T \boldsymbol{Nk\varepsilon_0}\,\mathrm{d}\Gamma,\ \overline{\boldsymbol{u_\Gamma}} = \int_{\Gamma_u^e} \boldsymbol{Z}^T \boldsymbol{u_\Gamma}\,\mathrm{d}\Gamma,\ \overline{\boldsymbol{u_{\Gamma_0}}} = \int_{\Gamma_u^e} \boldsymbol{Z}^T \boldsymbol{u_0}\,\mathrm{d}\Gamma, \end{aligned} \tag{10}$$

where $\widehat{(\cdot)}$ denotes a complex conjugate. The second set of the finite element system of equations (9) is formed by weak imposition of the Dirichlet boundary condition; the traction basis $\boldsymbol{Z}$ is chosen as the weighting matrix. Note that the system was derived for a single element. When multiple elements are connected, also the inter-element continuity conditions need to be imposed, which is performed in a similar way as the enforcement of the Dirichlet boundary condition. Due to the special choice of the basis functions, all the system matrices are constructed based on the integration along the element boundary, which allows to use elements of arbitrary shape.

4. Example: Comparison to Analytical Solution

The basis functions contained in the displacement approximation matrix $\boldsymbol{U}$ can directly be considered as the analytical solution to which the approximated one is compared. The Hankel function of the first kind and order $n = 4$ was chosen as the function W and only the part derived from the solenoidal potential is considered. Applying the kinematic and material equations, the associated stress field is derived and subsequently the boundary tractions can be recovered. When the problem is modelled using the hybrid Trefftz method, these tractions are applied as the boundary condition. The difference between average potential and kinetic energies E associated to the deformed state was chosen for assessment of the quality of the approximated results. Its approximation obtained using the hybrid Trefftz method is denoted by E_{FE}.

The structure is analysed using two finite element meshes, which discretize the domain into two and four elements. For each mesh, simulations for various orders N and M, which determine the number of terms contained in the displacement and traction bases, are performed. The Bessel function of the first kind is chosen as the function W appearing in the definition of the individual basis functions. The domain shape was chosen as the quarter of a hollow circle. The convergence plots are visualized in Figs. 1 and 2. For each traction basis order M, the number of terms contained in the displacement basis is increased. Therefore the individual lines illustrate an increase in the number of terms contained in the domain displacement basis while the length of the traction basis is fixed. The total number of degrees of freedom is plotted on the x-axis while the ratio E_{FE}/E is plotted on the y-axis. In Fig. 1, the results for the two-element mesh are

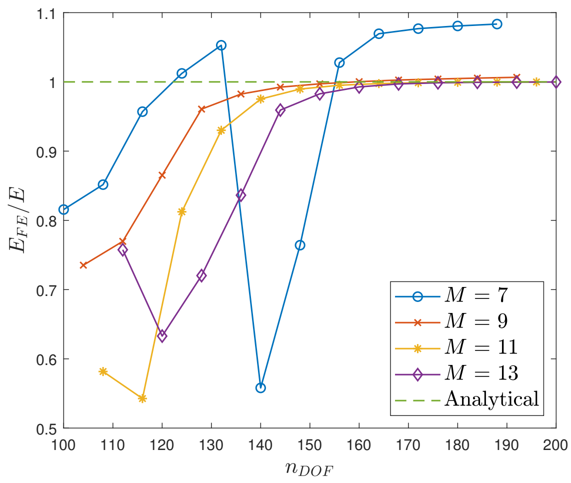

Fig. 1: Convergence: 2 elements

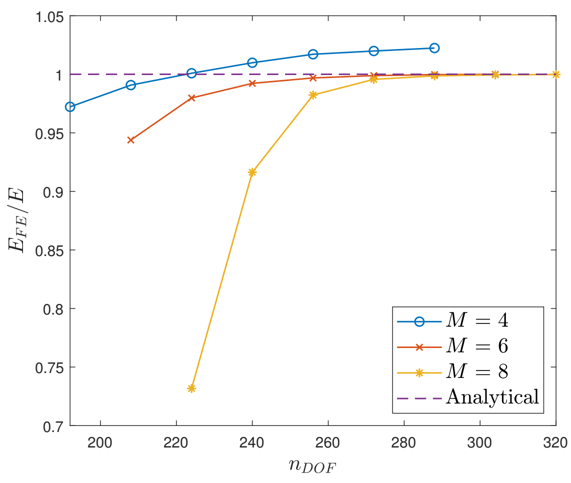

Fig. 2: Convergence: 4 elements

visualized, maximum orders M of the boundary traction basis from 7 to 13 and orders N of the domain basis from 10 to 21 were considered. Results for the four-element mesh are shown in Fig. 2. The boundary traction basis of maximum orders M from 4 to 8 was selected while the domain displacement basis orders vary from 9 to 15. From both figures it can be seen that for most of the considered lengths of the boundary traction basis the approximated results tend towards the reference solution, when the number of terms in the domain approximation basis is increased. However, for the two-element mesh when $M = 7$ and four-element mesh when $M = 4$ this statement does not hold any more and the results converge to a value which is significantly different compared to the analytically evaluated one. The reason for such mismatch is the fact that the true inter-element tractions cannot be captured sufficiently well by the polynomial of the mentioned orders and instead of refining the domain displacement basis the number of terms in the traction basis needs to be increased. As can be seen, the total number of degrees of freedom required for a certain accuracy is larger for the finer mesh, which motivates to model the analysed domain using only a few elements but bases with many terms. A drawback of such approach are the numerical difficulties which result in a badly conditioned system of equations.

5. Conclusions

Due to the fact that the approximation functions reflect the mechanical features of the modelled phenomenon, the domain can be discretized into only a few elements. To obtain more accurate results, the number of terms included in the domain basis is increased instead of refining the element mesh. Such p-refinement technique proves to produce equation systems with a relatively low number of degrees of freedom compared to conventional methods. Moreover, the resulting matrices appearing in the final system of equations are constructed by integration along the element boundary. As a consequence, elements of arbitrary shape and number of edges may be used. From the obtained results it can be concluded that the quality of the approximated solution is determined by the number of basis functions included in both the domain displacement as well as in the boundary traction basis. The maximum order of the polynomial contained in the traction basis needs to be high enough so that the shape of the true inter-element and boundary traction fields is well captured. If this condition is not fulfilled, even for an increasing number of terms contained in the domain displacement basis the approximated displacement solution does not converge to the true one. Overall, the hybrid Trefftz method represents an efficient solution procedure and offers some significant advantages compared to other deterministic approaches. Nevertheless, one has to be aware of the limitations of its application, since for domains of complex shapes the preferable efficiency is compromised.

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