29th International Conference
ENGINEERING MECHANICS 2023
Milovy, Czech Republic, May 9 – 11, 2023

THE CONNECTION BETWEEN THE PRINCIPLE OF THE LEAST ACTION AND THE THERMODYNAMIC CONDITION OF STABILITY

Maršík F.*, Trávníček Z.†, Antošová Z.‡

Abstract: *The direct correlation between classical mechanics of material points (Lagrange principle of classical mechanics) and classical continuum mechanics can be established when the existence of a trajectory and a friction force are added. The important role of the total enthalpy follows from a variational analysis. Moreover, the thermodynamic criterion of the stability is formulated using the total enthalpy and compared with the experiments, numerical and the classical Rayleigh theory supports its applicability. It was shown that the solid body vortex is at the margin of stability, which is experimentally observed. Analogously, the potential vortex is by the thermodynamic criterion stable, however by the Rayleigh criteria it is on the onset of stability. The loss of stability of the forsed vortex (solid body vortex) is the main reason why it transforms into a free vortex (potential vortex). The classical Taylor experiment of flow between two rotating cylinders is analyzed from the point of view of this criterion. Recently, the vortex transformation process has been demonstrated both experimentally and by numerical simulations for the case of a vortex tube at the Institute of Aerospace Thermodynamics at Stuttgart (Seibold, 2022) and experimentally for the annular nozzle flow at the Institute of Thermomechanics CAS in Prague.*

Keywords: Principle of the least action, Thermodynamic stability condition, Annular swirl flow, Vortex tube.

1. Introduction

In the mechanics of mass points (MP) in the case of conservative force fields (in the so-called Hamiltonian mechanics), the principle of least action is sufficient to describe their time evolution. Continuum mechanics can also be formulated using the extremal principle, but the physical meaning of the corresponding Lagrangian is extended both by the internal energy of the corresponding material point (MP) and by contact interaction (friction) with surrounding material points. Formulated in this way, the variational principle of continuum mechanics shows that the total enthalpy reaches its extreme value (minimum) even for processes with convection (Seliger, Witham , 1968).

2. Classical mechanics of continuous mechanical systems

The most general formulation of the laws governing the motion of all mechanical systems composed from many interacting particles is so-called *the principle of the least action* or the *Hamilton principle* (Seliger, Witham , 1968). In the application to the continuous system, see Fig. 1, we can evaluate the action $\mathcal{S}$ in the fix volume V with the surface ∂V between the instants t_0, t_1 as follows

$$\begin{aligned}\mathcal{S}\left(\mathbf{v},\boldsymbol{\beta},\mathbf{X}\right) &= \int_{t_o}^{t_1}\int_V \rho\left[\frac{\mathbf{v}^2(\mathbf{x},t)}{2} - \Phi(\mathbf{x}) - u\left(\rho(\mathbf{x},t), s(\mathbf{x},t)\right) - \mathbf{X}(\mathbf{x},\mathbf{t})\dot{\boldsymbol{\beta}}(\mathbf{x},\mathbf{t})\right] dvdt \\ &= \int_{t_o}^{t_1}\int_V \rho l(\mathbf{v}(\mathbf{x},t),\boldsymbol{\beta}(\mathbf{x},t),\mathbf{X}(\mathbf{x},t))dvdt \qquad (1)\end{aligned}$$

Specific lagrangian is $l(\mathbf{v}(\mathbf{x},t),\boldsymbol{\beta}(\mathbf{x},t),\mathbf{X}(\mathbf{x},\mathbf{t}))$.

* Prof. Ing. František Maršík, DrSc.: Institute of Thermomechanics AS CR, v.v.i.,Dolejškova 5, 18200 Prague, CZ marsik@it.cas.cz

† Assoc. Prof. Zdeněk Trávníček, PhD.: Institute of Thermomechanics AS CR, v.v.i.,Dolejškova 5, 18200 Prague, CZ tr@it.cas.cz

‡ Ing. Zuzana Antošová, PhD.: Institute of Thermomechanics AS CR, v.v.i.,Dolejškova 5, 18200 Prague, CZ antosova@it.cas.cz

ISSN: 1805-8256, doi: 10.21495/em2023-159

The standard abbreviation $\dot{()}$ is used for material derivative and further, the kinetic energy of material point is $\frac{\mathbf{v}^2}{2}$ and Φ is the potential energy, $\mathbf{X}\dot{\boldsymbol{\beta}} = \sum_{L=1}^{3} X^L \dot{\beta}_L = x^i \dot{\beta}_i(\mathbf{x}, t)$ is the energy of interaction (friction) with surroundings and $u\left(\rho((\mathbf{x}, t)), s((\mathbf{x}, t))\right)$ is the internal energy of the material point (M.P.). This variational principle (1) can also be applied to rigid bodies when internal energy $u(\mathbf{e}, t)$, depends on the Euler deformation tensor $\mathbf{e}(\mathbf{x}, \mathbf{t})$. The independent quantity is the trajectory $\mathbf{x}(\mathbf{X}, t)$ of the M.P. $\mathbf{X}$ and the velocity $\mathbf{v}(\mathbf{x}, t)$ and friction force $\boldsymbol{\beta}(\mathbf{x}, t)$ satisfy supplementary conditions. We suppose that each material point has the initial position $\mathbf{X}$-at some instant t_0. Its motion is described by the trajectory $\mathbf{x}(\mathbf{X}, t)$ (with the inverse mapping $\mathbf{X}\mathbf{x} = \mathbf{x}\mathbf{X} = \mathbf{I}$) and the motion of all material points materializes in the velocity field $\mathbf{v}(\mathbf{x}, t)$.

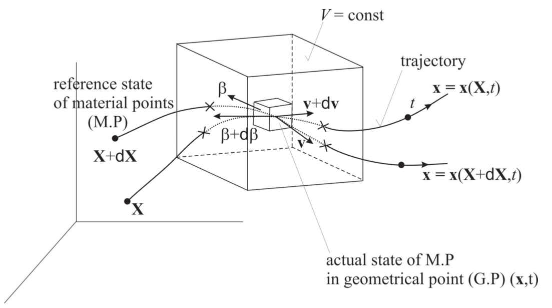

Fig. 1: At the G.P. $(\mathbf{x}, t)$ has the M.P. $\mathbf{X}$ the velocity field $\mathbf{v}\,(\mathbf{x}, t)$ and its interaction force is $\dot{\boldsymbol{\beta}}(\mathbf{x}, t)$.

2.1. Necessary conditions of the least action

Material point $\mathbf{X} = \tilde{\mathbf{X}}(\mathbf{x}, t)$ intersect the geometrical point (G.P.) $\mathbf{x}$ and interacts (exchanges the energy) with the surroundings due to friction force $\dot{\boldsymbol{\beta}}$. The general laws of mechanics and thermodynamics have to be valid for all material points, i.e. for all trajectories which intersect this point $\mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X}, t)$. So that the necessary condition of the extremum of the functional (1), with respect to the variations (fluctuations) $\delta\tilde{x}^i(\mathbf{X}, t) = \tilde{x}^i(\mathbf{X}, t) - \tilde{x}^i_0(\mathbf{X}, t)$, of the M.P. trajectory are as follows

$$\begin{aligned} \delta\mathcal{S} &= \int_{t_o}^{t}\int_V (\rho\delta l + l\delta\rho)\, dv dt = \int_{t_o}^{t}\int_V \left\{ \rho\left[-\dot{\beta}_L \frac{\partial \tilde{X}^L}{\partial x^i} - \frac{\partial\Phi}{\partial x^i} - T\frac{\partial s}{\partial x^i}\right]\delta\tilde{x}^i + \left(l - \frac{p}{\rho}\right)\frac{\partial\rho}{\partial x^i}\delta\tilde{x}^i \right. \\ &+ \left. \rho\left(v_i - X^L\frac{\partial\beta_L}{\partial x^i}\right)\frac{\partial\delta\tilde{x}^i}{\partial t} + \left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i}\right] X^L\delta\beta_L + \rho\left(\frac{\partial X^L}{\partial t} + v^i\frac{\partial X^L}{\partial x^i}\right)\delta\beta_L \right\} dv dt \\ &- \int_V \rho X^L \delta\beta_L dv\Big|_{t_o}^{t} - \int_{t_o}^{t}\int_{\partial V} \rho v^i x^l \delta\beta_L da_i dt = 0, \end{aligned} \tag{2}$$

where $\tilde{x}^i_0(\mathbf{X}, t)$ is unknown optimum (reference) trajectory, which satisfy extremum condition (2). Velocity variation $\delta v^i = \delta\dot{\tilde{x}}^i$ depends on the $\delta\tilde{x}^i(\mathbf{X}, t)$. The extremum conditions relevant for our flow stability problem are

$$v_i = X^L\frac{\partial\beta_L}{\partial x^i} = \frac{\partial\,\overbrace{(X^L\beta_L)}^{\varphi(\mathbf{x},t)}}{\partial x^i} - \beta_L\frac{\partial X^L}{\partial x^i} = v_{i\,\text{pot}} + v_{i\,\text{rot}}, \quad v_{i\,\text{pot}} = \frac{\partial\varphi}{\partial x^i},\; v_{i\,\text{rot}} = \beta_i = -\beta_L\frac{\partial X^L}{\partial x^i} \tag{3}$$

The vorticity **rot v** is generated by friction or by the entropy gradient, as it will be shown later and it can induce the instability. The laws of conservation of the energy in the G.P. is

$$l - \frac{p}{\rho} = -X^L\frac{\partial\beta_L}{\partial t} - h_t = 0 \quad \text{...energy conservation,} \quad \text{for} \quad h_t = \frac{v^2}{2} + u + \frac{p}{\rho} + \Phi \tag{4}$$

and the static pressure $p(\mathbf{x}, t)$ is equal to Lagrangian density ρl (1). ***Total enthalpy*** h_t ***depends on the local friction force, only*** . For the local stationary friction field, i.e., $\partial(\beta)/\partial t = 0$, the total enthalpy of the fluid is constant. From point of view of classical mechanics of mass points, the total enthalpy is the ***integral of motion*** and h_t can be taken as an ***adiabatic invariant***. The balance of momentum can be reformulated in the measurable quantities as follows

$$-\dot{\beta}_L\frac{\partial\tilde{X}^L}{\partial x^i} - \frac{\partial\Phi}{\partial x^i} - T\frac{\partial s}{\partial x^i} = 0 \;\; \text{elimination of } \boldsymbol{\beta},\; \frac{\partial v_i}{\partial t} - (\mathbf{v}\times\mathbf{rot}\,\mathbf{v})_i = -\frac{\partial h_t}{\partial x^i} + T\frac{\partial s}{\partial x^i} + \frac{\partial\Phi}{\partial x^i} \tag{5}$$

$$\text{or} \quad -(\mathbf{v}\times\mathbf{rot}\;\mathbf{v})_i = -\frac{\partial h_t}{\partial x^i} + T\frac{\partial s}{\partial x^i} + \frac{\partial t^{li}_{dis}}{\rho\partial x^l} \quad \text{Crocco theorem-for steady state only} \tag{6}$$

The friction (dissipative) forces in actual system are established explicitly by the dissipative part of the stress tensor $\mathbf{t}_{dis}$, which represents the effect of surface forces and $s(\mathbf{x}, t)$ is the entropy. Nevertheless, even for isentropic flow with h_t=0, the volume force can induce the vortex generation.

3. Thermodynamic stability conditions

Definition of entropy S follows from the global form of the ***II. Law of thermodynamics*** formulated as the balance of the total entropy

$$\dot{S} - J_D(S) = P(S) \geq 0, \quad \text{for} \quad P(S) = \int_{\mathcal{V}} \sigma(s) d\nu \geq 0 \tag{7}$$

where $J_D(S)$ is entropy flux, $P(S) \geq 0$ is total entropy production and $\sigma(S) \geq 0$ is the density of entropy production. The thermodynamic stability criterion can be formulated by h_t, see (Maršík, 1999). The entropy is the function of h_t, p and the thermodynamic inequality (II. Law of thermodynamics) is

$$\pi = T\sigma(S) = \rho \underbrace{(T\dot{s} + \frac{1}{\rho}\frac{\partial p}{\partial t} - \dot{h}_t)}_{=0\ldots\text{entropy definition}} - \frac{q^k}{T}\frac{\partial T}{\partial x^k} + \frac{\partial(t^{ki}_{dis} v_i)}{\partial x^k} \geq 0, \quad -\frac{\rho}{2}\dot{\overline{d^2 h_t}} = \pi, \tag{8}$$

for $\pi < 0$ is the thermodynamic inequality violated and the instability can occurs. The inequality (8) is *the thermodynamic condition for the stability of the process*.

4. Consequences of thermodynamic stability conditions.

For the simplified flow, $v_\varphi = v_\varphi(r)$, $T = T(r)$ has the form

$$\tilde{\pi} = \frac{\lambda}{T}\left(\frac{\partial T}{\partial r}\right)^2 + \mu\left[\left(\frac{\partial v_\varphi}{\partial r}\right)^2 + v_\varphi \frac{\partial^2 v_\varphi}{\partial r^2} - \frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial r}\right] \geq 0 \tag{9}$$

Considering that the thermal conductivity is a positive coefficient $\lambda > 0$ the associated term is always positive and will only have a stabilizing effect. We therefore focus only on the influence of the flow field $v_\varphi(r)$, where the viscosity μ plays dominant role.

$$\begin{aligned} v_\varphi &= \omega r \quad \text{then} \quad \tilde{\pi} = 0 \quad \text{solid body vortex with angular velocity } \omega \\ v_\varphi &= \Gamma / r \quad \text{then} \quad \tilde{\pi} = 4\Gamma^2/r^4 \geq 0 \quad \text{potential vortex} \end{aligned} \tag{10}$$

The solid body vortex is on the onset of stability and the potential vortex with circulation Γ is stable.

4.1. Couette flow

The stability condition

$$\frac{\tilde{\pi}}{\mu_{mol}} = \frac{4\Omega_1^2 \eta^4 (1-\mu)^2}{\tilde{r}^4 (1-\eta^2)^2} \geq 0 \tag{11}$$

can also be interpreted in the following way

$$\frac{\tilde{\pi}}{\mu_{mol}} = \frac{4\Omega_1^2 \eta^4}{\tilde{r}^4} > 0 \tag{12}$$

$$\begin{aligned} \text{for} \ (1-\mu)^2 &= (1-\eta^2)^2 \\ \text{or} \ \Omega_1 &= \Omega_2 \left(\frac{R_1}{R_2}\right)^{-2} \ \text{and} \end{aligned}$$

$$\Omega_1 = \Omega_2 \left(2 - \left(\frac{R_1}{R_2}\right)^2\right)^{-1} \tag{13}$$

The onset of instability is given by the black straight lines $\Omega_1 = 1.292\Omega_2$, for $\Omega_2 > 0$- (right) and for $-\Omega_2 \in (-250, 0)$ by $\Omega_1 = \frac{-\Omega_2}{\left(\frac{R_1}{R_2}\right)^2 - 2} = 0.815\Omega_2$-(left). Considering that for a given ratio of R_1/R_2, which is always less than 1, the inner cylinder can rotate even faster than the outer one, see Fig. 2.

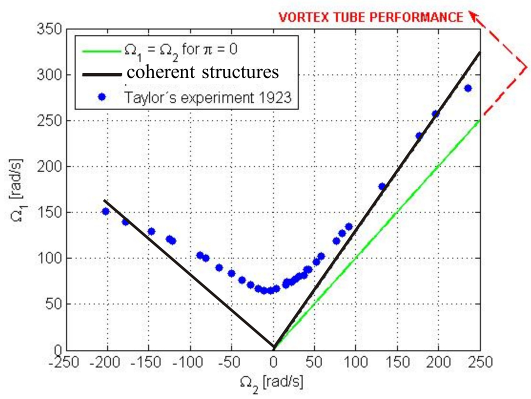

Fig. 2: Onset of coherent structures (instability) for a viscous flow between two rotating cylinders for $\eta = R_1/R_2 = 0.8798$ (Taylor-Couette flow) (Taylor, 1923).

The thermodynamic stability criterion of the process (9), which respects the existence of viscosity, supplements the purely mechanical Rayleigh criterion with an additional condition that can be interpreted as the onset of coherent structures (Taylor-Couette flow).

4.2. Annular swirling jet experiments

The scheme of the present setup is shown in Fig. 3. The settling chamber located upstream at the nozzle operated as the swirl generator. For this purpose, the chamber was supplied with axial (main) and swirling (tangential, control) air flow inlets with the mass fluxes m_A and m_S, respectively.

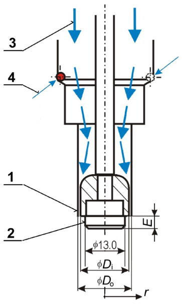

Fig. 3: The annular nozzle and tested configuration. 1: outer nozzle body, 2: nozzle centerbody, 3: axial air flow supply, 4: swirling air flow supply through a pair of tangential ports (diameter 3.1 mm); H: nozzle-to-wall spacing, E: centerbody extension. Dimensions: $D_0 = 17.6\,\text{mm}$, $D_i = 15.85\,\text{mm}$, *and* $E = 4.1\,\text{mm}$.

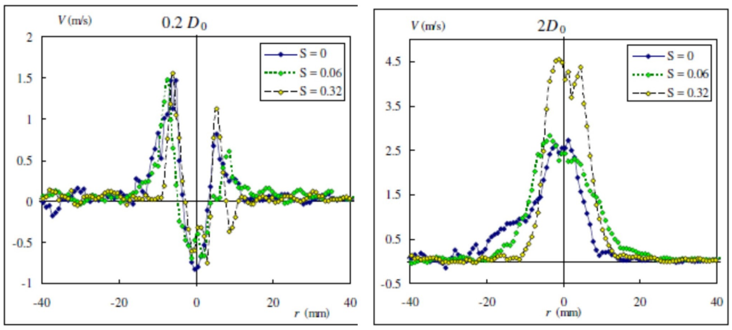

Fig. 4: PIV measurement with cross-stream velocity profiles and streamwise velocity component at different distances measured from the bottom part of the nozzle centerbody (0.2-2)D_0*. The experiments were performed at* $\text{Re} = 5000$ *for five variants of the jets depending on the swirl number S from 0 to 0.32*

The experiment in Fig. 4 shows how the annular current forms into a potential vortex. If the viscosity of the fluid is non-zero, the potential vortex is more stable under the given conditions than the solid body vortex, see thermodynamic stability condition (10).

5. Conclusions

To underline it the thermodynamic criteria of stability of the steady state and the stability of the processes are applied to two specific cases. These findings can be satisfactorily unified by using the properties of total enthalpy. The important role of the total enthalpy for inviscid flow follows from the variational analysis (Seliger, Witham , 1968). The use of total enthalpy offers a somewhat more general view on the stability of flow of viscous fluids.

Acknowledgments

We gratefully acknowledge the support of the Grant Agency of the Czech Republic-Czech Science Foundation (Project No. 21-26232J) and the institutional support RVO: 61388998.

References

Maršík F. (1999), *Continuum Thermodynamics*. Praha: Academia, (in Czech, Termodynamika kontinua).

Seibold F. Ligrani P. and Weigand B. (2022), Flow and heat transfer in swirl tubes –A review. *International Journal of Heat and Mass Transfer* **187 122455, pp. 1-26**.

Seliger, R.L. Whitham, G.B. (1968), Variational Principles of Continuum mechanics, *Proc. Roy Soc. A.305*, **1-25**. See https://www.jstor.org/stable/2416171

Taylor G.I. (1923). Stability of a viscous liquid contained between two rotating cylinders. *Philosophical Transactions of the Royal Society of London A*, **223, pp. 289 - 343**. See https://doi.org/10.1098/rsta.1923.0008