30th International Conference
ENGINEERING MECHANICS 2024
Milovy, Czech Republic, May 14 – 16, 2024

STOCHASTIC VERSION OF THE ARC-LENGTH METHOD

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Abstract: *The solution of a nonlinear algebraic system using the incremental method, based on pre-defined loading steps, fails in the vicinity of local extrema as well as around bifurcation points. The solution involved the derivation of the so-called 'Arc-Length' method. Its essence lies in not incrementing the system parameter or any of the independent variables but rather the length of the response curve. The stochastic variant of this method allows for working with a system where system parameters include random imperfections. This contribution presents a variant that tracks the first two stochastic moments. Even in this simple case, interesting phenomena can be observed, such as the disappearance of the energy barrier against equilibrium jump due to random imperfections in the system.*

Keywords: Random imperfection, stochastic arc-length method, continuation, numerical method.

1. Introduction

In the study of slender and flexible structures, the amplitudes of deflections or vibrations, and consequently the corresponding deformations, can reach values that cannot be approximately considered linear. Mathematical models must account for the arising nonlinearity, introducing significant complications in mathematical processing. Finding the equilibrium state requires solving nonlinear systems. There can exist multiple equilibrium states, which in themselves can be stable or unstable. Transitions between them occur spontaneously or through the breakthrough of energy barriers. Such a process may be entirely local and insignificant for the overall system, allowing its loading to continue, or, alternatively, it may signify partial or complete collapse of the system.

The general formulation of the problem in static analysis reads:

$$\mathbf{F}(\mathbf{r}, \mathbf{u}, \lambda) = 0\,, \tag{1}$$

where $\mathbf{r}$ represents the internal parameters of the system (describing geometry, physical properties, etc.) with m components; $\mathbf{u}$ is the displacement vector of nodes in the network with n components; λ is the load parameter. The function $\mathbf{F}$ is assumed as sufficiently continuous.

A similar concept can be applied when seeking the stationary response of dynamic systems. If some form of averaging is employed, it is possible to formulate the corresponding differential equation in slow time, where the characteristic feature for the stationary solution is a zero differential part.

The solution of the system (1) using the incremental method with pre-defined loading steps $\Delta\lambda$, see for example (Jagannathan et al., 1975; Bergan, 1980), fails around the extremum of any of the curves $u^i(\lambda)$ and further in the vicinity of a bifurcation point. After many attempts to find a solution, each with only a limited range of applicability, a more universal method was proposed by Riks (1979), which was named the "Arc-Length" method. Its essence lies in not incrementing λ or any of the components of $\mathbf{u}$ but rather the length of the response curve arc. The length of the vector $|\Delta\mathbf{u}^t, \Delta\lambda|$ is specified, assuming that it does not differ

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doi: 10.21495/em2024-210

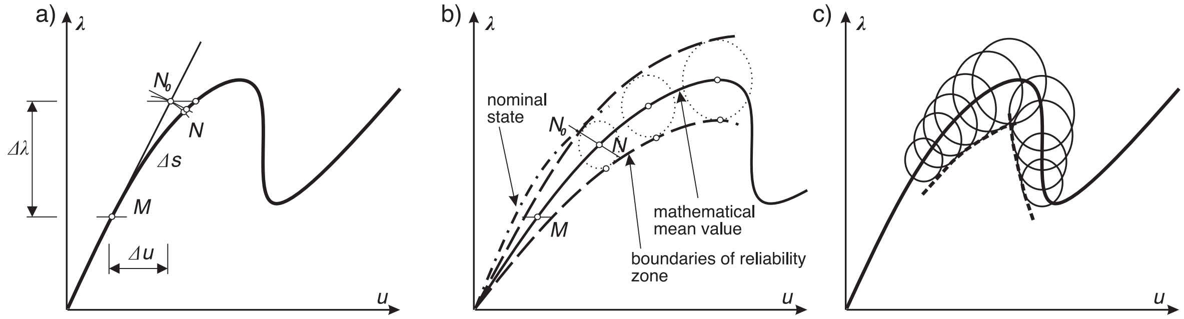

Fig. 1: a) Incremental and Arc-Length method, b) stochastic Arc-Length method, c) mathematical expectation and reliability zone.

significantly from the actual length of the arc as a curve. Thus, the increment of the load parameter is one of the unknowns, rather than a predetermined quantity.

It turns out that the fundamental idea of the Arc-Length method can be extended to the domain of systems with random imperfections, as shown by Náprstek (1994, 1999).

2. The stochastic arc-length method

In the stochastic case, the perturbed parameters $\mathbf{r}$ should be understood as the sum of the deterministic and stochastic component:

$$\mathbf{r} = \mathbf{r}_d + \mathbf{r}_\varepsilon \,, \tag{2}$$

where $\mathbf{r}_d$ represents the nominal values of system parameters, and $\mathbf{r}_\varepsilon$ denotes parameter perturbations as Gaussian random processes in suitable coordinates. The system (1) thus transforms into a system of stochastic equations. Its solution will be formulated in weak stochastic sense.

Let $\partial_{\mathbf{u}}\mathbf{F}^t(\mathbf{r}, \mathbf{u}_{N_j}, \lambda_{N_j})$ denote the incremental matrices in the j-th approximation step to the point N, where the coordinates statistically sufficiently satisfy the system (1). This point is approached from the initial point M by incrementing the arc length by Δs, as depicted in Fig. 1a:

$$\partial_{\mathbf{u}}\mathbf{F}^t(\mathbf{r}, \mathbf{u}_{N_j}, \lambda_{N_j}) = \mathbf{C}^t_{N_j} \quad ; \quad \partial_\lambda \mathbf{F}(\mathbf{r}, \mathbf{u}_{N_j}, \lambda_{N_j}) = \mathbf{\Lambda}_{N_j} \,. \tag{3}$$

Matrices $\mathbf{C}N_j$ and $\mathbf{\Lambda}N_j$ describing the local stiffness state at point N_j are influenced by imperfections $\mathbf{r}_\varepsilon$. The linear stochastic approximation of parameter deviations from the nominal state can be introduced as linear combinations of suitable deterministic basis functions. The coefficients of this linear combination are random processes. The basis functions can encompass, for example, systems of cones with vertices at individual nodes of the mesh in case of discretized system, values of Fourier coefficients representing the expansion of imperfect surfaces, etc. Consequently, imperfections are modeled as m random scalar processes entering into (1) or (2). This implies that the local stiffness matrices (3) can be expressed in the form

$$\mathbf{C}_{N_j} = \mathbf{C}^0_{N_j} + \sum_{i=1}^{m} \mathbf{C}^i_{N_j} r^i_\varepsilon \quad ; \quad \mathbf{\Lambda}_{N_j} = \mathbf{\Lambda}^0_{N_j} + \sum_{i=1}^{m} \mathbf{\Lambda}^i_{N_j} r^i_\varepsilon \,. \tag{4}$$

where $\mathbf{C}^0_{N_j},, \mathbf{\Lambda}^0_{N_j}$ represent the local stiffness matrices at point N_j of the system in the nominal state, $\mathbf{C}^i_{N_j}$, $\mathbf{\Lambda}^i_{N_j}$ denote the increments of matrices $\mathbf{C}^0_{N_j}$, $\mathbf{\Lambda}^0_{N_j}$ due to a "unit" imperfection, and r^i_ε is the value of the i-th imperfection (a Gaussian-centered random process).

The increments of displacements and loads are expressed in a similar form, i.e., as linear combinations of certain vectors, where the coefficients of these combinations are the same random processes r^i_ε as in the case of (4):

$$\Delta\mathbf{u}_{N_j} = \Delta\mathbf{u}^0_{N_j} + \sum_{i=1}^{m} \Delta\mathbf{u}^i_{N_j} r^i_\varepsilon \quad ; \quad \Delta\lambda_{N_j} = \Delta\lambda^0_{N_j} + \sum_{i=1}^{m} \Delta\lambda^i_{N_j} r^i_\varepsilon \,. \tag{5}$$

The solution of the response of the imperfect system is based on the incremental form of the system (1) supplemented by the constant arc-length condition $\Delta s = \text{const}$. This means that the increment of the load is unknown; it is one of the unknowns, just like all the increments of displacements at the nodes. The advance from point M to point N on the response hyperplane proceeds in two steps, as shown in Fig. 1b.

The initial step involves moving in the tangential direction towards point N_0. This is accomplished by solving a linear system that is generated as follows: starting with the incremental operation (1), subsequent substitutions as per (3), (4), (5) (where point N_j is substituted with point M), and then applying the mathematical expectation operator $\mathcal{E}\cdot$. This process yields the first part of the solution; the part which describes the mathematical expectation and generalized coordinates of the stochastic component of the response during the first step from point M to point N_0:

$$\begin{gathered}\mathbf{C}_M^0 \Delta\mathbf{u}_M^0 + \mathbf{\Lambda}_M^0 \Delta\lambda_M^0 + \sum_{i=1}^{m}\sum_{k=1}^{m}\left(\mathbf{C}_M^i \Delta\mathbf{u}_M^k + \mathbf{\Lambda}_M^i \Delta\lambda_M^k\right) K^{ik} = 0\,, \\ \Delta\mathbf{u}_{M^-}^{0t} \Delta\mathbf{u}_M^0 + \Delta\lambda_{M^-}^0 \Delta\lambda_M^0 + \sum_{i=1}^{m}\sum_{k=1}^{m}\left(\Delta\mathbf{u}_{M^-}^{it} \Delta\mathbf{u}_M^k + \Delta\lambda_{M^-}^i \Delta\lambda_M^k\right) K^{ik} = \Delta s^{(2)}\,,\end{gathered} \tag{6}$$

where it was used:

$$\mathsf{E}\{r_\varepsilon^i \cdot r_\varepsilon^k\} = K^{ik} \quad ; \quad \mathsf{E}\{r_\varepsilon^i\} = 0\,. \tag{7}$$

The second step of the calculation aims to reach point N on the response curve through successive iterations, with the goal of achieving the best possible accuracy. This is done to ensure the following equations are optimally satisfied:

$$\mathsf{E}\{\mathbf{F}(\mathbf{u}_N, \lambda_N)\} = 0 \quad ; \quad \mathsf{E}\{r_\varepsilon^i \cdot \mathbf{F}(\mathbf{u}_N, \lambda_N)\} = 0\,. \tag{8}$$

In the j-th step, that is, at point N_j, however, these equations are not satisfied, and the following holds:

$$\mathsf{E}\{\mathbf{F}(\mathbf{u}_{N_j}, \lambda_{N_j})\} = \mathbf{\Phi}_{N_j}\ ; \quad \mathsf{E}\{r_\varepsilon^i \cdot \mathbf{F}(\mathbf{u}_{N_j}, \lambda_{N_j})\} = \mathbf{\Psi}_{N_j}^i\ ; \quad i = 1,..,m\ ;\ j = 0,1,.. \tag{9}$$

The iteration will take place in the hyperplane perpendicular to the previous tangent (line MN_0) of the surface (1), or along the hypersphere with the center at point M (depending on whether the linear or quadratic version of the Arc-Length method is used). Similar steps as in the first step deliver the algebraic system of $(1+m)(n+1)$ equations for the unknown increments $\Delta\mathbf{u}_{N_j}^0$, $\Delta\lambda_{N_j}^0$, $\Delta\mathbf{u}_{N_j}^k$, $\Delta\lambda_{N_j}^k$ $(k = 1, \dots, m)$:

$$\begin{gathered}\mathbf{C}_{N_j}^0 \Delta\mathbf{u}_{N_{j+1}}^0 + \mathbf{\Lambda}_{N_j}^0 \cdot \Delta\lambda_{N_{j+1}}^0 + \sum_{i=1}^{m}\sum_{k=1}^{m}\left(\mathbf{C}_{N_j}^i \cdot \Delta\mathbf{u}_{N_{j+1}}^k + \mathbf{\Lambda}_{N_j}^i \cdot \Delta\lambda_{N_{j+1}}^k\right) K^{ik} = -\mathbf{\Phi}_{N_j}\,, \\ \Delta\mathbf{u}_{N_j}^{0t} \Delta\mathbf{u}_{N_{j+1}}^0 + \Delta\lambda_{N_j}^0 \Delta\lambda_{N_{j+1}}^0 + \sum_{i=1}^{m}\sum_{k=1}^{m}\left(\Delta\mathbf{u}_{N_j}^{it} \Delta\mathbf{u}_{N_{j+1}}^k + \Delta\lambda_{N_j}^i \Delta\lambda_{N_{j+1}}^k\right) K^{ik} = 0\,,\end{gathered} \tag{10}$$

$$\begin{gathered}\sum_{i=1}^{m}\left(\mathbf{C}_{N_j}^i \Delta\mathbf{u}_{N_{j+1}}^0 + \mathbf{\Lambda}_{N_j}^i \Delta\lambda_{N_{j+1}}^0\right) K^{li} + \sum_{k=1}^{m}\left(\mathbf{C}_{N_j}^0 \Delta\mathbf{u}_{N_{j+1}}^k + \mathbf{\Lambda}_{N_j}^0 \Delta\lambda_{N_{j+1}}^k\right) K^{lk} = -\mathbf{\Psi}_{N_j}^l\,, \\ \sum_{i=1}^{m}\left(\Delta\mathbf{u}_{N_j}^{it} \Delta\mathbf{u}_{N_{j+1}}^0 + \Delta\lambda_{N_j}^i \Delta\lambda_{N_{j+1}}^0\right) K^{li} + \sum_{k=1}^{m}\left(\Delta\mathbf{u}_{N_j}^{0t} \Delta\mathbf{u}_{N_{j+1}}^k + \Delta\lambda_{N_j}^0 \Delta\lambda_{N_{j+1}}^k\right) K^{lk} = 0\,.\end{gathered} \tag{11}$$

Based on these calculations, the overall response of the system during one step of the Arc-Length method from point M to point N can be described as

$$\mathbf{u}_N = \mathbf{u}_M + \sum_{j=0}^{\mu}\left(\Delta\mathbf{u}_{N_j}^0 + \sum_{i=1}^{m} \Delta\mathbf{u}_{N_j}^i r_\varepsilon^i\right)\,. \tag{12}$$

Hence, the value of the mathematical expectation and mutual correlation of the individual response component at point N read

$$\mathsf{E}\{\mathbf{u}_N\} = \mathbf{u}_M + \sum_{j=0}^{\mu} \Delta\mathbf{u}^0_{N_j}\,, \tag{13}$$

$$\mathsf{E}\{\mathbf{u}_N \cdot \mathbf{u}^t_N\} - \mathsf{E}\{\mathbf{u}_N\} \cdot \mathcal{E}\{\mathbf{u}^t_N\} = \sum_{k=0}^{\mu}\sum_{j=0}^{\mu}\left(\sum_{i=1}^{m}\sum_{l=1}^{m} \Delta\mathbf{u}^i_{N_j}\Delta\mathbf{u}^{lt}_{N_k} \cdot K^{il}\right)\,. \tag{14}$$

Formulas (13), (14) enable tracing the curves of the most probable response (mathematical mean) and the corresponding dispersion zone. Properties (e.g. diameter) of this zone indicates how much the critical load level needs to be reduced due to introduced imperfections.

The outcomes of the entire calculation can be understood with reference to Fig. 1c. For every point along the curve of the mathematical mean response, there is a corresponding curve depicting the probability density distribution of deviations from the mathematical mean at individual nodes or degrees of freedom. Given that the analysis was confined to the first two stochastic moments, each of these curves represents a Gaussian distribution with a variance derived from the second-moment calculation.

Consequently, upper and lower bounds are established, forming a region on both sides of the curve which represents the most probable response. This region is known as the reliability region. It ensures that, with the specified probability and imperfection statistics, the system's response will not fall outside this defined area. The process of deriving both upper and lower bound curves can result in a complex profile, as suggested in Fig. 1c. This complexity indicates that certain sections of these curves may not be relevant for the original purpose. Both upper and lower curves may exhibit inflection points and various special properties, necessitating evaluation methods that exploit the particular characteristics of their geometry.

3. Concluding remarks

Tthe stochastic Arc-Length method is a powerful tool for addressing the challenges posed by uncertainties and nonlinearities in structural analysis, providing insights into the reliability of structures under stochastic conditions. It uses the arc length as a parameter to trace the response of the structure. This allows for efficient tracking of nonlinear responses, especially in situations where traditional methods may encounter convergence issues. While the method often focuses on the first few stochastic moments, higher moments may be considered for a more comprehensive stochastic analysis.

Acknowledgment

The kind support of Czech Science Foundation project No. 24-13061S and of the RVO 68378297 institutional support are gratefully acknowledged.

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